Problem Set 12

For this entire problem set, $R = k[x_1, x_2, \ldots, x_n]$ with k a field.

Recall that $R = k[x_1, x_2, \ldots, x_n]$ is a graded ring by the decomposition $R = \bigoplus_{d \geq 0} R_d$ where R_d is the space spanned by the monomials of degree d. Given a graded R-module, M, we can define the Hilbert function ϕ_M of M by $\phi_M(t) = dim_k(M_t)$ for each $t \in \mathbb{Z}$. We have the following theorem:

Theorem 1. *(Hilbert-Serre) Let M be a finitely generated graded $R = k[x_0, x_1, \ldots, x_n]$ module. There is a unique polynomial $P_M(t) \in \mathbb{Q}[t]$ such that $\phi_M(t) = P_M(t)$ for $t \gg 0$.*

The polynomial that appears in the theorem above is called the Hilbert Polynomial of M. In the near future, we will have tools that will enable us to compute the Hilbert Polynomial. For now, we must be satisfied to compute the Hilbert Function.

For now I will let dimension be defined intuitively, the empty set will have dimension -1, points will have dimension 0, curves will have dimension 1, etc. Later we will give a more precise definition of dimension.

Recall the following lemma from Problem Set 9:

Lemma 2. *Let $R = k[x_1, x_2, \ldots, x_n]$. Let $F \in R$. There is a short exact sequence of the form*

$$0 \to R/(I : F) \to R/I \to R/(I, F) \to 0.$$

If we assume the $deg(F) = d$, then we can make the exact sequence respect the grading by noting that

$$0 \to (R/(I : F))_t \xrightarrow{\times F} (R/I)_{t+d} \to (R/(I, F))_{t+d} \to 0.$$

This might be helpful in the following problems.

Problem 3. *a) Let $I = (x^3, y^2, z)$. Compute the Hilbert Function of $k[x, y, z]/I$.*

b) What is the dimension of $V(I)$ as a set in $\mathbb{P}^2$?

Problem 4. *a) Let P_1, P_2, P_3 be your favorite 3 non-collinear points in $\mathbb{P}^2$. Let I be the homogeneous ideal of the union of these 3 points. Compute the Hilbert Function of R/I.*

b) What is the dimension of $V(I)$?

Problem 5. *Let I be the homogeneous ideal of the 3-uple embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$. Compute the Hilbert Function of R/I.*

b) What is the dimension of $V(I)$?

Problem 6. *a) Do you notice any pattern in the previous 3 problems?*

b) Test your guess on R/I where I is the 2-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$.

Let $V \subset \mathbb{P}^2$ be a smooth planar curve defined as the zeroes of the irreducible homogeneous polynomial $F \in k[x, y, z]$. The tangent line to V at P is defined to be the zeroes of $xF_x(P) + yF_y(P) + zF_z(P)$. Define a map $\phi : V \to (\mathbb{P}^2)^*$ by $\phi(P) =$ the dual of the tangent line to V at P. V^* is defined to be the image of this map.

If $F \in k[x, y, z]$ and $F^* \in k[X, Y, Z]$ then we can compute F^* by $F^* = (X - F_x, Y - F_y, Z - F_z, F) \cap k[X, Y, Z]$ (you should think about why this is true!).

The singularities of F can be determined by looking for singularities on each of the 3 affine pieces (or by noting that the singularities occur at the points where $F_x = F_y = F_z = 0$).

Problem 7. *a) Let $V = V(x^2 + 3yz + y^2)$. Is V^* singular?*

b) Find the equation which defines V^*.

c) Is V^* singular?

Problem 8. *a) Let $V = V(x^3 + 3xyz + y^2z + z^3)$. Is V singular?*

b) Find the equation which defines V^*.

c) Is V^* singular?

Problem 9. *If V is non-singular but V^* is singular, what might be the source of these singularities? (Think geometrically!)*

Problem 10. *Try computing V^{**} for a couple of examples. Does it seem $V = V^{**}$?*

Problem 11. *Do you see how you could define V^* if V is a surface in $\mathbb{P}^3$?*