Problem Set 13

Problem 1. *It is a well known theorem that the diagonals of a parallelogram bisect each other. If A, B, C, D are the vertices of the parallelogram and if M is the point where the diagonals intersect then the theorem can be shown to be true by showing that the condition $(M_1 - C_1)^2 + (M_2 - C_2)^2 - (M_1 - B_1)^2 - (M_2 - B_2)^2 = 0$ (and a similar equation for the other diagonal) is forced by the assumptions of the problem. This is essentially an ideal membership question.*

a) The following conditions of the problem are clear: $\overline{AB} \parallel \overline{CD}$, $\overline{AC} \parallel \overline{BD}$, A, M, D are collinear, B, M, C are collinear. Are any other conditions needed?

b) Now put all the conditions on the computer and prove the theorem computationally.

Pascal's Theorem says the following:

Theorem 2. *Let C be an irreducible conic. Let $P_1, P_2, \ldots, P_6$ be 6 points on the conic. Let A be the intersection of $\overline{P_1P_2}$ and $\overline{P_4P_5}$, B be the intersection of $\overline{P_2P_3}$ and $\overline{P_5P_6}$, and C be the intersection of $\overline{P_3P_4}$ and $\overline{P_6P_1}$. Then A, B, C are collinear.*

Problem 3. *Prove Pascal's theorem computationally.*

Let $G \in k[x_1, x_2, \ldots, x_{n+1}]$ be a homogeneous polynomial. Let $V = V(G) \subseteq \mathbb{P}^n$. The next problem is used to establish that the singularities of V are the zeroes of the ideal $(G_{x_1}, G_{x_2}, \ldots, G_{x_{n+1}}, G)$. In other words, you don't have to look at each affine piece individually, you can make a single calculation that includes all of the affine pieces.

Problem 4. *Let $F \in k[x_1, x_2, \ldots, x_n]$ and $F^h \in k[x_1, x_2, \ldots, x_{n+1}]$. Let $P = (a_1, a_2, \ldots, a_n) \in V(F) \subseteq \mathbb{A}^n$. Then $P^h = [a_1 : a_2 : \cdots : a_n : 1] \in V(F^h) \subseteq \mathbb{P}^n$. Show that $F_{x_1}(P) = F_{x_2}(P) = \cdots = F_{x_n}(P) = 0 \iff (F^h)_{x_1}(P^h) = (F^h)_{x_2}(P^h) = \cdots = (F^h)_{x_{n+1}}(P^h) = 0$.*

Let $F, G \in R = k[x, y, z]$ be two polynomials which do not share common factors. Let $P \in V(F, G)$. The intersection number of F and G at P is defined to be $I(P, F \cap G) = dim_k(\mathcal{O}_P(\mathbb{A}^2)/(F, G))$. If $P = (a, b)$ then let $I_P = (x - a, y - b)$. Intersection numbers can be determined computationally as $\lim_{n \to \infty} dim_k(R/(I_P^n, F, G))$. In a practical way, this can be determined by increasing n until $dim_k(R/(I_P^n, F, G))$ stabilizes.

Problem 5. *Let $F = y^2 - x^3 - x^2$ and $G = (x^2 + y^2)^2 + 3x^2y - y^3$. Let $P = (0, 0)$. Compute $I(P, F \cap G)$.*