Problem Set 14

A quick note on degree. Let V be an affine variety of dimension p. If you intersect V with a "random" hyperplane then you will get a variety of dimension $p-1$. If you intersect V with p random hyperplanes then you should get something which is zero dimensional. On the level of ideals, if you take the union of $I(V)$ with the ideal of p random linear forms then you will get an ideal J. The **degree** of V is defined to be $dim_k(R/J)$. We can compute the **dimension** of V by determining how many hyperplane sections are needed to reduce V to a set of points.

Suppose V is a smooth irreducible curve in $\mathbb{A}^3$. To each point $P \in V$ we can glue the tangent line to V at P. In this manner, we get a surface in $\mathbb{A}^3$ called the tangent variety of V. Here is one strategy for computing the equation of such a surface from $I(V)$. Let $I(V) \subseteq k[x,y,z]$. Construct an ideal J in $k[A,B,C,x,y,z]$ that contains all of the information of all the tangent lines at each choice of $(A,B,C) \in V$. This can be done by the following steps:

1) Construct the Jacobian matrix, $Jac(I)$, and multiply it by the transpose of the matrix $[x-A, y-B, z-C]$. Take all of the entries of the resulting product and form an ideal L.

2) Form the ideal I' by taking the generators of I and replacing x with A, y with B, and z with C.

3) Now the ideal generated by $L + I'$ contains all of the information we need. Compute $G = (L + I') \cap k[x,y,z]$. G is the ideal of the tangent variety.

Problem 1. *Let* $I = (x^2 - y, x^3 - z) \subseteq k[x,y,z]$. *Then* I *is the ideal of the twisted cubic curve in* $\mathbb{A}^3$. *The tangent variety of the twisted cubic is a surface and will be generated by one equation,* F.

a) What is the degree of the twisted cubic?

b) What do you think the degree of F *will be?*

c) Compute F. *Were you correct?*

d) Can you gain any feeling for the relationship of the degree of F *to the degree of* V*?*

Problem 2. *Repeat the previous problem for the Veronese surface in* $\mathbb{A}^5$.