Problem Set 16

In an earlier problem set, it was shown that any three points in $\mathbb{A}^2$ can be moved to any other set by an affine change of coordinates. The next problem is a similar problem in $\mathbb{P}^n$. Let $V \subseteq \mathbb{P}^n$ be a set of points. V is said to be in **linearly general position** if any subset of V of size $n+1$ does not lie on a hyperplane and if any subset of V of size $k \leq n$ does not lie on a $k-1$-plane.

Problem 1. *Prove that any 2 ordered sets of $n+2$ points in linearly general position in $\mathbb{P}^n$ are projectively equivalent.*

Problem 2. *Can you find a condition that allows you to determine when 4 points in $\mathbb{P}^1$ are projectively equivalent? Hint: Let the two sets be $\{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$. Move P_1 to 1, move P_2 to ∞, and move P_3 to 0. Where does P_4 get moved?*

A modification of this procedure can be used to determine when $n+3$ ordered points in $\mathbb{P}^n$ are linearly equivalent (see pages 7,8,12 of Harris). An explicit answer is not known for determining when two ordered sets of more than $n+3$ points are linearly equivalent. Work on it!

Problem 3. *Show that a line through 2 flexes on a cubic passes through a third flex.*

The next three problems are from Fulton's book on algebraic curves.

Problem 4. *Let C be the non singular cubic curve in $\mathbb{P}^2$ determined by the equation $y^2x = x^3 + ay^2z + bxz^2 + cz^3$. Let $0 = [0:1:0]$. Let $P_i = [x_i : y_i : 1]$ for $i = 1, 2, 3$. Suppose $P_1 \oplus P_2 = P_3$. If $x_1 \neq x_2$ then let $\lambda = (y_1 - y_2)/(x_1 - x_2)$. If $P_1 = P_2$ and $y_1 \neq 0$, let $\lambda = (3x_1^2 + 2ax_1 + b)/(2y_1)$. Let $\mu = y_i - \lambda x_i$ (for $i = 1$ or 2). Show that $x_3 = \lambda^2 - a - x_1 - x_2$ and $y_3 = \lambda x_3 - \mu$.*

Problem 5. *Let $C = V(y^2z - x^3 - 4xz^2), 0 = [0:1:0], A = [0:0:1], B = [2:4:1], C = [2:-4:1]$. Show that $0, A, B, C$ form a subgroup of order 4 and that it is cyclic.*

Problem 6. *Let $C = V(y^2z - x^3 - 43xz^2 - 166z^3), 0 = [0:1:0], P = [3:8:1]$. Show that P is an element of order 7 in C.*