Problem Set 18

Problem 1. *a) Let $R = k[x, y, z]$. Show that every syzygy of the matrix $\begin{bmatrix} x & y \end{bmatrix}$ is of the form $A \begin{bmatrix} -y \\ x \end{bmatrix}$ where $A \in R$.*

b) Show that every syzygy of $\begin{bmatrix} x & y & z \end{bmatrix}$ is an R-linear combination of the three syzygies $\begin{bmatrix} -y \\ x \\ 0 \end{bmatrix}, \begin{bmatrix} -z \\ 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ -z \\ y \end{bmatrix}$.

Problem 2. *Let $R = k[x_1, x_2, \ldots, x_n]$. Show that every syzygy of $\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$ is generated by an R-linear combination of the $\binom{n}{2}$ syzygies of the form*

$$\begin{bmatrix} 0 & 0 & \ldots & -x_i & \ldots & x_j & 0 & \ldots & 0 \end{bmatrix}^T$$

where $-x_i$ is in the j^{th} spot and x_j is in the i^{th} spot and there are zeroes in every other spot.

Problem 3. *Let Q_1, Q_2 be two general quadrics in $R = k[w, x, y, z]$. Let $I = (Q_1, Q_2)$. It can be shown that the minimal free resolution of R/I is*

$$0 \to R(-4) \to R(-2)^2 \to R \to R/I \to 0.$$

Use this free resolution to compute the Hilbert polynomial of R/I.

Problem 4. *Let Q_1, Q_2, Q_3 be three general quadrics in $R = k[w, x, y, z]$. Let $I = (Q_1, Q_2, Q_3)$. It can be shown that the minimal free resolution of R/I is*

$$0 \to R(-6) \to R(-4)^3 \to R(-2)^3 \to R \to R/I \to 0.$$

Use this free resolution to compute the Hilbert polynomial of R/I.

Problem 5. *Let $F = x^3 + x^2y + z^3 \in k[x, y, z]$. Let $S = k[X, Y, Z]$. Let elements in S act on F by partial differentiation.*

a) Find $I_S(F)$.

b) Determine the Hilbert function of $S/I_S(F)$.