Problem Set 19

Problem 1. *Let $G = Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 \in \mathbb{C}[x, y, z]$. You can quickly determine that*

$$\begin{bmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix}.$$

Show that G is irreducible if and only if the 3×3 matrix that appears above has rank three.

Problem 2. *Let $R = k[x, y, z]$ and let $S = k[X, Y, Z]$. Let elements of S act on elements of R by partial differentiation. Let $J = (6YZ - 5Z^2, 6Y^2 - 4Z^2, 6XZ - 3Z^2, 6XY - 2Z^2, 6X^2 - Z^2) \subseteq S$. Find an $F \in R$ such that $J = I_S(F)$.*

Let I be the ideal generated by the maximal minors of an $r \times (r+t)$ matrix. If $V(I)$ has codimension $t+1$ then explicit resolutions have been determined for I. These varieties have been well studied and form the easiest case of *Determinantal Varieties.* You can find the free resolutions of such varieties in Eisenbud's book. A special case of such a matrix is the $r \times (r+k)$ matrix:

$$\begin{bmatrix} x_0 & x_1 & \dots & x_k & 0 & 0 & 0 & \dots & 0 \\ 0 & x_0 & x_1 & \dots & x_k & 0 & 0 & \dots & 0 \\ 0 & 0 & x_0 & x_1 & \dots & x_k & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & x_0 & x_1 & \dots & x_k \end{bmatrix}.$$

If I is the ideal generated by the maximal minors of this matrix then $I = (x_0, x_1, \dots, x_k)^r$ and the Hilbert function of R/I can be determined by many methods. The Hilbert function of R/I where I comes from a general $r \times (r+k)$ matrix of linear forms will be the same as this one.

Problem 3. *Let M be an $r \times (r+1)$ matrix of linear forms in $R = k[x_0, x_1, \dots, x_n]$. Let I be the ideal generated by the $r \times r$ minors of M. If $V(I)$ has codimension 2 then we can determine explicitly a free resolution of I and use it to compute the Hilbert polynomial of R/I. Alternatively, we can compute the Hilbert polynomial using the comments above.*

a) Let $n = 2$. Find the Hilbert polynomial of R/I as a function of r in 2 different ways and check that your answers are the same.

b) Repeat part a) with $n = 3$.