Problem Set 20

Problem 1. *Let F and G be two homogeneous polynomials in $R = k[x_0, x_1, \dots, x_n]$. Assume that F and G do not share any common factors. Suppose that $A, B \in R$ and $AF + BG = 0$. Prove that there exists an $H \in R$ such that $A = -HG$ and $B = HF$. (In other words prove that every syzygy of $\begin{bmatrix} F & G \end{bmatrix}$ is of the form $H \begin{bmatrix} -G \\ F \end{bmatrix}$.)*

Suppose the Hilbert Polynomial of R/I is $F(X) = a_r X^r + a_{r-1}X^{r-1} + \cdots + a_1 X + a_0$. Associated to I is a scheme (not necessarily equidimensional). The dimension of the scheme is equal to r. The degree of the scheme is $a_r/r!$. If $F(X) = a_1 X + a_0$ then the scheme is one dimensional and has degree a_1. Let $G = 1 - a_0$. If the scheme is one dimensional then G is called the arithmetic genus.

Problem 2. *a) Use the previous problem to determine the Hilbert Polynomial of $k[x_0, x_1, x_2, x_3]/(F, G)$ where F is irreducible of degree 3 and G is irreducible of degree 5.*

b) When do the Hilbert Function and Hilbert Polynomial of $k[x_0, x_1, x_2, x_3]/(F, G)$ start to agree?

c) What is the arithmetic genus of the scheme defined by F and G. (If F and G meet transversally at their points of intersection then we can think of the variety defined by F and G instead of the scheme defined by F and G.)

Problem 3. *Let M be a 4×4 matrix. Let M_{top} denote the top two rows of M. Let M_{bottom} denote the bottom two rows of M. It was claimed in class that the determinant of M could be written in terms of the 2×2 minors of M_{top} and the 2×2 minors of M_{bottom}. Let a_{ij} denote the ij^{th} entry of M. If $M^{ij} = a_{1i}a_{2j} - a_{2i}a_{1j}$ and $M_{ij} = a_{3i}a_{4j} - a_{4i}a_{3j}$, then write the determinant of M in terms of these symbols and verify that it is correct.*

Problem 4. *Recall that the join of two varieties $V, W \subseteq \mathbb{P}^n$ is defined to be the closure of the set of all lines joining a point of V to a point of W. We will write this as $J(V, W)$. Let $R = k[a, b, c, d, e]$. Let L_1 be the line defined by the ideal $(a - b, c + d, e) \subseteq R$ and let L_2 be the line defined by the ideal $(a + c, b + d, a + b + e) \subseteq R$. Find generators for the ideal of $J(L_1, L_2)$.*

Problem 5. *Determine the equation of the secant variety to the Veronese Surface in $\mathbb{P}^5$.*