Four bar linkage coupler curve

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Let $k = \mathbb{R}$. Consider the following problem:

Problem. Compute the equation of the *four bar linkage curve* obtained by taking the points $A = (0,0)$, $B = (0,2), C = (x_1, y_1)$ and $D = (x_2, y_2)$ in $\mathbb{A}^2_k$ (see Problem Set 7 for the definition).

To solve this problem, we use the elimination theory. Consider the following ideal in $k[x_1, x_2, y_1, y_2, X, Y]$:

$$\begin{aligned} I &= (x_1^2 + y_1^2 - 1, x_2^2 + (y_2 - 2)^2 - 1, (x_1 - x_2)^2 + (y_1 - y_2)^2 - 1, \\ &\quad (X - x_1)^2 + (Y - y_1)^2 - 4, (X - x_2)^2 + (Y - y_2)^2 - 4). \end{aligned}$$

Our task is to compute the ideal $I \cap k[x, y]$ in $k[x, y]$. To make this computation more effectively (in `Macaulay2`), we eliminate variables one by one. Use `Eliminate 1`. This is an optional argument of `MonomialOrder` that is the elimination order eliminating the first variable. To compute the Gröbner basis, use also `gb(I, Strategy=>Primary)`. This is a new (?) algorithm, that is often faster than the default algorithms.

```
i1 : KK=QQ;

i2 : S=KK[x_1,x_2,y_1,y_2,X,Y,MonomialOrder=>Eliminate 1];

i3 : I=ideal(x_1^2+y_1^2-1,x_2^2+(y_2-2)^2-1,(x_1-x_2)^2+(y_1-y_2)^2-1,(X-x_1)^2+(

o3 : Ideal of S

i4 : I1=ideal selectInSubring(1,gens gb(I,Strategy=>Primary));

o4 : Ideal of S
```

Define the new ring `S1`$= k[x_2, y_1, y_2, X, Y]$ and plug `I1` into `S1` to obtain the ideal `I1'` in `S1`. Then compute the Gröbner basis for `I1'`:

```
i5 : S1=KK[x_2,y_1,y_2,X,Y,MonomialOrder=>Eliminate 1];
```

```
i6 : I1'=substitute(I1,S1);

o6 : Ideal of S1

i7 : I2=ideal selectInSubring(1,gens gb(I1',Strategy=>Primary));

o7 : Ideal of S1
```

Iterate this operation until we eliminate the last variable y_2:

```
i8 : S2=KK[y_1,y_2,X,Y,MonomialOrder=>Eliminate 1];

i9 : I2'=substitute(I2,S2);

o9 : Ideal of S2

i10 : I3=ideal selectInSubring(1,gens gb(I2',Strategy=>Primary));

o10 : Ideal of S2

i11 : S3=KK[y_2,X,Y,MonomialOrder=>Eliminate 1];

i12 : I3'=substitute(I3,S3);

o12 : Ideal of S3

i13 : time I4=ideal selectInSubring(1,gens gb(I3',Strategy=>Primary));
     -- used 0.21 seconds

o13 : Ideal of S3

i14 : S4=KK[X,Y,MonomialOrder=>Lex];

i15 : I5=substitute(I4,S4)

                12     10 2      10        10       8 4       8 3      8 2       8
o15 = ideal(X   + 6X  Y  - 12X  Y - 38X   + 15X Y  - 60X Y  - 78X Y  + 276X Y + 57

o15 : Ideal of S4
```

The polynomial defines a curve C in $\mathbb{A}_k^2$. The four bar linkage coupler curve does not fill up this curve, because we can attach the triangle in two different ways and each time we get a different curve. So the ideal `I5` corresponds to the union of these two curves. Unfortunately, `I5` does not seem to be decomposable over $\mathbb{Q}$. Thus we cannot have the defining ideal for each curve. Do you have any idea about how to see these curves?

Surprisingly, `Maple` shows us both the components of C! This computer algebra system has the command `implicitplot`, that creates the two-dimensional plot of an implicitly defined curve (such as a circle). The following is the picture of the curves defined by the equation we obtained:

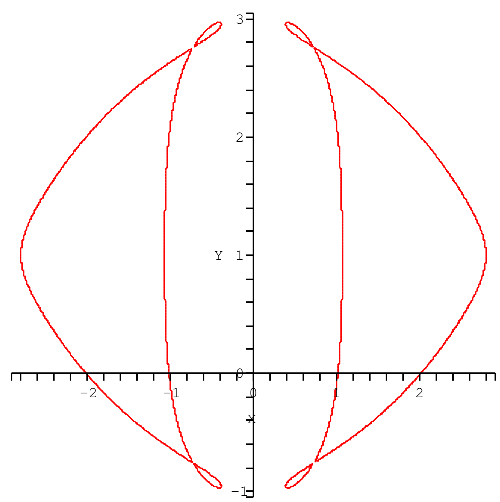

Figure 1: Two four bar linkage coupler curves.

Appendix. Here we make the same computation with SINGULAR, another computer algebra system for research in algebraic geometry, commutative algebra and singularity theory. (Please visit the SINGULAR homepage:

`http://www.singular.uni-kl.de`

if you want to know more about SINGULAR.) This computer algebra system is now available to use on Linux. Open a shell window by clicking `SSH Linux` on your desktop. Start SINGULAR with the command `Singular` in the shell:

```
% Singular

                     SINGULAR                             /
 A Computer Algebra System for Polynomial Computations   /   version 2-0-4
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   June 2003
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
```

SINGULAR has the command `elim` for elimination. This function takes an ideal I and two integers n, m as inputs and returns the ideal obtained from I by eliminating from the n variable up to the mth variable. Use the command `LIB` for loading the library `elim.lib`:

```
> LIB "elim.lib";
// ** loaded /usr/local/Singular/2-0-4/LIB/elim.lib (1.14.2.4,2003/04/16)
// ** loaded /usr/local/Singular/2-0-4/LIB/poly.lib (1.33.2.6,2003/02/10)
// ** loaded /usr/local/Singular/2-0-4/LIB/ring.lib (1.17.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-4/LIB/general.lib (1.38.2.9,2003/04/04)
// ** loaded /usr/local/Singular/2-0-4/LIB/matrix.lib (1.26.2.3,2003/05/14)
// ** loaded /usr/local/Singular/2-0-4/LIB/random.lib (1.16.2.1,2002/02/20)
// ** loaded /usr/local/Singular/2-0-4/LIB/inout.lib (1.21.2.5,2002/06/12)
```

Define the ring:

```
> ring R=0,(a,b,c,d,x,y),lp;
```

This means `ring name=coefficient-ring, name-of-variables, monomial-ordering`. In our case, the name of ring is `R`, the coefficient field is $\mathbb{Q}$, the variables are `a,b,c,d,x,y` and the monomial order `lp` is the lexicographic order. Define the ideal:

```
> ideal I=(a^2+b^2-1,c^2+(d-2)^2-1,(a-c)^2+(b-d)^2-1,(x-a)^2+(y-b)^2-4,(x-c)^2+(y-
```

Do elimination with `elim`:

```
> elim(I,1,4);
_[1]=x12+6x10y2+15x8y4+20x6y6+15x4y8+6x2y10+y12-12x10y-60x8y3-120x6y5-
120x4y7-60x2y9-12y11-38x10-78x8y2+68x6y4+292x4y6+258x2y8+74y10+
276x8y+528x6y3-72x4y5-624x2y7-300y9+571x8+1388x6y2+1762x4y4+1644x2y6+
699y8-4152x6y-8808x4y3-5160x2y5-504y7-3522x6-854x4y2+666x2y4-
2002y6+20412x4y+27768x2y3+7356y5+13473x4-6030x2y2-3119y4-55188x2y-
24468y3-30780x2+8964y2+43200y+20736
```