Homework 3 (Math461 EO)

Problem 1 (2 points)

Assume that $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{Z}$. Use this assumption and induction to prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for all integers $n \geq 2$ and arbitrary intergers $a_1, a_2, \ldots, a_n$.

Problem 2 (2 points)

Show that if the statement

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n$$

is assumed to be true for some n, then it can be proved to be true for $n + 1$. Is the statement true for all $n \in \mathbb{N}$?

Problem 3 (2 points)

Let $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$. Define a binary operation on $\mathbb{Z}_n$ by

$$[a][b] = [ab]$$

for all $[a], [b] \in \mathbb{Z}_n$. Prove that it is well-defined, i.e., prove that if $[a] = [a']$ and $[b] = [b']$, then $[a][b] = [a'][b']$.

Problem 4 (2 points)

In each of the following, a rule is given that determines a binary operation $*$ on $\mathbb{Z}$. Determine in each case whether $*$ is commutative or associative and whether there is an identity element.

(i) $x * y = x + y + 3$.

(ii) $x * y = x + xy$.

Problem 5 (2 points)

Let S be a set of three elements given by $S = \{A, B, C\}$. In the following table, all the elements of S are listed in a row at the top and in a column at the left. The result of $x * y$ is found in the row that starts with x at the left and in the column that has y at the top. For example, $B * C = C$ and $C * B = A$.

$*$	A	B	C
A	C	A	B
B	A	B	C
C	B	A	C

(i) Is the binary operation $*$ commutative? Why?

(ii) Determine whether there is an identity element in S with respect to $*$.

(iii) If there is an identity element, which elements have inverses?