Homework 4 (Math461 EO)

Problem 1 (2.5 points)

Let $S_3 = \{e, \sigma_1, \ldots, \sigma_5\}$ be the set of permutations on $\{1, 2, 3\}$, where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$
$$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Complete the multiplication table for S_3.

Note. You don't need to show your entire work. Please complete the table and describe just a few cases explicitly.

Problem 2 (2.5 points)

Let $A \in GL(n, \mathbb{R})$ and let $O(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) \mid A^T A = AA^T = I_n \}$. Prove that $O(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.

Hint. A^T denotes the transpose of A. This means that if $A = (a_{ij})_{1 \leq i,j \leq n}$, then $A^T = (a_{ji})_{1 \leq i,j \leq n}$. You may use the fact that $(A^T)^T = A$ and $(AB)^T = B^T A^T$.

Problem 3 (2.5 points)

Let G be a group and let H_1 and H_2 be subgroups of G. Prove that $H_1 \cap H_2$ is a subgroup of G. Do you think $H_1 \cup H_2$ is also a subgroup?

Problem 4 (2.5 points)

Decide whether each of the following sets is a subgroup of $G = \{1, -1, i, -i\}$ under multiplication. If a set is not a subgroup, give the reason why it is not.

(i) $\{1, -1\}$.

(ii) $\{1, i\}$.

(iii) $\{i, -1\}$.

(iv) $\{1, -i\}$.

Bonus problem (2 points) For a fixed element a of a group G, prove that the subset

$$C_a := \{ \ x \in G \mid ax = xa \ \}$$

of G is a subgroup of G. This subgroup is called the *centralizer* of a in G.