Sample Exam 3 (Math461 Fall 2009)

Problem 1 (Permutation groups)

Make each of the following true or false.

(i) Every groups G is a subgroup of S_G.

(ii) The symmetric group S_3 is cyclic.

(iii) S_n is not cyclic for any positive integer n.

Problem 2 (Permutation groups)

Find the order of (1 12 8 10 4)(2 13)(5 11 7)(6 9).

Problem 3 (Permutation groups)

Let A be a set, let B be a non-empty subset of A and let b be one particular element of B. Determine whether $\{\sigma \in S_A \mid \sigma(b) \in B\}$ is a subgroup of S_A.

Problem 4 (Homomorphisms)

Let $\phi : \mathbb{C}^\times \to \mathbb{R}^\times$ be the map defined by $\phi(z) = |z| = \sqrt{z\bar{z}}$ for all $z \in \mathbb{C}^\times$. Prove that ϕ is a homomorphism, and determine the kernel of ϕ.

Problem 5 (Homomorphisms)

Let G and G' be groups. Assume that $\phi : G \to G'$ is a homomorphism. Let K be a subgroup of G'. Prove that $\phi^{-1}(K) = \{x \in G \mid \phi(x) \in K\}$ is a subgroup of G.

Problem 6 (Isomorphisms)

Let G be a group. For each element a in G, define a map $k_a : G \to G$ by $k_a(x) = xa^{-1}$ for all x in G.

(i) Prove that each k_a is a permutation on the set of elements of G.

(ii) Prove that $K = \{k_a | a \in G\}$ is a group with respect to map composition.

(iii) Define $\phi : G \to K$ by $\phi(a) = k_a$ for each a in G. Determine wether ϕ is always an isomorphism.

Problem 7 (Automprohisms)

Let G be an arbitrary group. Prove or disprove that the map $\phi(a) = a^{-1}$ is an automorphism of G.

Problem 8 (Automorphisms)

Suppose that $\gcd(m, n) = 1$ and let $\phi : \mathbb{Z}_n \to \mathbb{Z}_n$ be defined by $\phi([a]) = m[a]$. Prove or disprove that ϕ is an automorphism.

Problem 9 (Isomorphisms)

Let H be the subset of $\mathrm{GL}(2, \mathbb{R})$ defined by

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$$

Prove that the additive group $\mathbb{Z}$ is isomorphic to H.

Problem 10 (Cosets)

Consider the set of matrices $G = \{I_2, A_1, A_2, A_3, A_4, A_5\}$, where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \; A_4 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \; A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These matrices form a group whose multiplication table is the following:

$\cdot$	I_2	A_1	A_2	A_3	A_4	A_5
I_2	I_2	A_1	A_2	A_3	A_4	A_5
A_1	A_1	I_2	A_4	A_5	A_2	A_3
A_2	A_2	A_5	A_3	I_2	A_1	A_4
A_3	A_3	A_4	I_2	A_2	A_5	A_1
A_4	A_4	A_3	A_5	A_1	I_2	A_2
A_5	A_5	A_2	A_1	A_4	A_3	I_2

Let $H = \{I_2, A_2, A_3\}$. Then it follows from the above multiplication table that H is a subgroup of G.

(a) Prove that H is a normal subgroup of G.

(b) Find the index $[G : H]$.