Sample Final (Math461 Fall 2009)

Problem 1 (Factor groups)

Let $G = S_3$. For each H that follows, show that the set of all left cosets of H in G does not form a group with respect to a product defined by $(aH)(bH) = (ab)H$.

(i) $H = \{e, (12)\}$.

(ii) $H = \{e, (13)\}$.

(iii) $H = \{e, (23)\}$.

Problem 2 (Factor groups)

Consider the matrices

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{R})$. Let $G = \{I_2, R, R^2, R^3, H, D, V, T\}$, where I_2 is the 2×2 identity matrix. Then G is a group with respect to matrix multiplication. Define a map $\phi : G \to G'$ by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

where $G' = \{\pm 1\}$ under multiplication.

(i) Prove that ϕ is an onto homomorphism.

(ii) Find the elements of $\ker(\phi)$.

(iii) Write out the distinct elements of the factor group $G/\ker(\phi)$ of G modulo $\ker(\phi)$.

Problem 3 (Factor groups)

Let G be a group.

(i) Prove that the center $Z(G)$ is a normal subgroup of G.

(ii) Prove that if $G/Z(G)$ is cyclic, then G is abelian.

Problem 4 (Isomorphism theorem)

Let G be an arbitrary group and let H and K be normal subgroups of G with $K \leq H$.

(i) Prove that H/K is a normal subgroup of G/K.

(ii) Prove that $(G/K)/(H/K)$ is isomorphic to G/H.

Problem 5 (External direct products)

In (i) through (iii), find the orders of the element of the direct product.

(i) $([2],[6]) \in \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$.

(ii) $([8],[10]) \in \mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$.

(iii) $([3],[10],[9]) \in \mathbb{Z}_4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{15}$.

Problem 6 (External direct products)

How many elements of order 4 does $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ have?

Problem 7 (External direct products)

Find all order-2 subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Problem 8 (External direct products)

Let G be a group, let e be the identity of G and let H and K be normal subgroups of G such that $H \cap K = \{e\}$. Prove that G is isomorphic to a subgroup of $G/H \oplus G/K$.

Problem 9 (Fundamental theorem of finite abelian groups)

Find all abelian groups, up to isomorphism, of order 32.

Problem 10 (Fundamental theorem of finite abelian groups)

How many abelian groups (up to isomorphism) are there of order 24? of order 25, of order $24 \cdot 25$?

Problem 11 (Fundamental theorem of finite abelian groups)

Let $G = U(\mathbb{Z}_{32})$ and let $H = \{[1], [31]\} \leq G$.

(i) Show that $|G| = 16$.

(ii) The factor group G/H of G modulo H is isomorphic to one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one.