Homework 6 (Math462)

Problem 1 (4 points)

Assume that

$$R = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{Z} \right\}$$

is a ring with respect to matrix addition and multiplication.

(i) Let $\varphi : R \to \mathbb{Z}$ be the map defined by

$$\varphi \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} = x.$$

Prove that φ is an onto ring homomorphism.

(ii) Describe $\ker(\varphi)$ and exhibit an isomorphism from $R/\ker(\varphi)$ to $\mathbb{Z}$.

Problem 2 (2.5 points)

Let I and J be ideals of a ring R. Prove that $I/(I \cap J)$ is isomorphic to $(I + J)/J$.

Hint. Define a map $\varphi : I \to (I + J)/J$ by $\phi(x) = x + J$. Prove that φ is an onto ring homomorphism and that $\ker(\varphi) = I \cap J$. Then use the first isomorphism theorem to prove that $I/(I \cap J)$ is isomorphic to $(I + J)/J$,

Problem 3 (2.5 points)

Assume that

$$R = \left\{ \begin{pmatrix} m & 2n \\ n & m \end{pmatrix} \,\middle|\, m, n \in \mathbb{Z} \right\}$$

and

$$R' = \{m + n\sqrt{2} \mid m, n \in \mathbb{Z}\}$$

are rings with respect to their usual operations. Prove that R and R' are isomorphic.

Problem 4 (2.5 points)

Let R and R' be rings with unities 1_R and $1_{R'}$ respectively. Prove that if $\phi : R \to R'$ is an onto ring homomorphism, then $\phi(1_R) = 1_{R'}$.