Sample Exam 1 (Math462 Spring 2009)

Problem 1 (Division Algorithm)

For $f(x)$, $g(x)$, and $\mathbb{Z}_p[x]$, given in (i) and (ii), find the quotient $q(x)$ and remainder $r(x)$ when dividing $f(x)$ by $g(x)$.

(i) $f(x) = 2x^5 + 2x^4 + x^2 + 2$, $g(x) = x^3 + 2x^2 + 2 \in \mathbb{Z}_3[x]$.

(ii) $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$, $g(x) = 3x^2 + 2 \in \mathbb{Z}_7[x]$.

Problem 2 (Division Algorithm)

Let $F = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_p$ with p prime. Prove that if $f(x)$ and $g(x)$ are non-zero polynomials of $F[x]$ such that $f(x)|g(x)$ and $g(x)|f(x)$, then there exits a non-zero $c \in F$ such that $f(x) = cg(x)$.

Problem 3 (Euclidean Algorithm)

For $f(x)$, $g(x)$, and $\mathbb{Z}_p[x]$, given in (i) and (ii), find $s(x)$, $t(x) \in \mathbb{Z}_p[x]$ such that $\gcd(f(x), g(x)) = s(x)f(x) + t(x)g(x)$.

(i) $f(x) = x^4 + 2x^2 + x + 1$, $g(x) = 2x^3 + 2x^2 + x + 1 \in \mathbb{Z}_3[x]$.

(ii) $f(x) = x^4 + 5x^2 + 2x + 2$, $g(x) = 3x^2 + 2 \in \mathbb{Z}_7[x]$.

Problem 4 (Factor Rings)

Let $f(x) = 3x^5 - 4x^2 \in \mathbb{Z}_5[x]$ and let $I = \{s(x)f(x) \mid s(x) \in \mathbb{Z}_5[x]\}$.

(i) Prove or disprove that every non-zero element of $\mathbb{Z}_5[x]/I$ has an inverse.

(ii) Find the dimension of $\mathbb{Z}_5[x]/I$.

Problem 5 (Rings)

Define a new binary operation of addition on $\mathbb{Z}$ by $x \oplus y = x + y - 1$ and a new binary operation of multiplication on $\mathbb{Z}$ by $x \otimes y = x + y - xy$. Do you think $\mathbb{Z}$ forms a ring with respect to these binary operations?

Problem 6 (Rings)

Let U be a non-empty set and let $\mathcal{P}(U)$ be the power set of U (i.e., the set of all subsets of U). For arbitrary A, $B \in \mathcal{P}(U)$, we define $A + B$ and AB by

$$\begin{aligned} A + B &= (A \cup B) \setminus (A \cap B) \\ &\quad \{x \in U \mid x \in A \text{ or } B \text{ and } x \text{ is not in both } A \text{ and } B\} \end{aligned}$$

and $AB = A \cap B$ respectively. Prove that $\mathcal{P}(U)$ is a ring with respect to these binary operations.

Note. You may assume that addition is associative.

Problem 7 (Rings)

Let R and S be rings. In the Cartesian product $R \times S = \{(r, s) \mid r \in R \text{ and } s \in S\}$, define

$$\begin{aligned} (r_1, s_1) + (r_2, s_2) &= (r_1 + r_2, s_1 + s_2) \\ (r_1, s_1)(r_2, s_2) &= (r_1 r_2, s_1 s_2) \end{aligned}$$

Prove that $R \times S$ is a ring with respect to these binary operations.

Note. This group is called the *direct sum* of R and S and denoted by $R \oplus S$.

Problem 8 (Rings)

Suppose that R is a ring in which all elements x satisfy $x^2 = x$.

(i) Prove that $x = -x$ for each $x \in R$.

(ii) Prove that R is commutative.

Problem 9 (Subrings)

Let R_1 and R_2 be subrings of a ring R. Prove that $R_1 \cap R_2$ is again a subring of R. Do you think $R_1 \cup R_2$ is always a subring of R?

Problem 10 (Subrings)

For a fixed element a of a ring R, prove that the set $\{x \in R \mid ax = 0\}$ is a subring of R.