Sample Exam 3 (Math462 Sring 2010)

Problem 1 (Mixed)

Make each of the following true or false.

(i) A polynomial $f(x)$ of degree n with coefficients in a field F can have at most n zeros in F.

(ii) Every polynomial of degree 1 in $F[x]$ has at least one zero in the field F.

(iii) Every field is a UFD.

(iv) Every UFD is a PID.

(v) If D is a UFD, then $D[x]$ is a UFD.

(vi) $\mathbb{C}$ is a simple extension of $\mathbb{R}$.

(vii) $\mathbb{Q}$ is an extension of $\mathbb{Z}_2$.

(viii) Every non-constant polynomial in $F[x]$ has a zero in some extension field of F.

Problem 2 (Irreducibility)

Demonstrate that $f(x) = x^4 + 2x^2 + 8x + 1 \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$.

Problem 3 (Irreducibility)

Let $f(x) = x^3 + 6 \in \mathbb{Z}_7[x]$. Write $f(x)$ as a product of irreducible polynomials over $\mathbb{Z}_7$.

Problem 4 (Irreducibility)

Let p be a prime integer and consider the polynomials $f(x) = x^p$ and $g(x) = x$ over $\mathbb{Z}_p$. Prove that $f(c) = g(c)$ for all c in $\mathbb{Z}_p$.

Problem 5 (Irreducibility)

Find the number of irreducible monic quadratic polynomials in $\mathbb{Z}_p[x]$, where p is a prime.

Problem 6 (UFD)

Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

Problem 7 (Prime elements and irreducible elements)

Prove that if p is irreducible in a UFD, then p is a prime.

Problem 8 (Fields)

Let F and F' be fields and let $\varphi : F \to F'$ be a ring homomorphism. Prove that either φ is the zero map or φ is one-to-one.