Sample Final Exam (Math462 Sring 2009)

Problem 1 (Mixed)

Make each of the following true or false.

(i) Every field is a UFD.

(ii) Every UFD is a PID.

(iii) If D is a UFD, then $D[x]$ is a UFD.

(iv) $\mathbb{C}$ is a simple extension of $\mathbb{R}$.

(v) $\mathbb{Q}$ is an extension of $\mathbb{Z}_2$.

(vi) Every non-constant polynomial in $F[x]$ has a zero in some extension field of F.

(vii) Every finite extension of a field is an algebraic extension.

(viii) Every algebraic extension of a field is a finite extension.

Problem 2 (UFD)

Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

Problem 3 (Prime elements and irreducible elements)

Prove that if p is irreducible in a UFD, then p is a prime.

Problem 4 (Fields)

Let F and F' be fields and let $\varphi : F \to F'$ be a ring homomorphism. Prove that either φ is the zero map or φ is one-to-one.

Problem 5 (Field extensions)

Let $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$.

(i) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.

(ii) Let α be a zero of $f(x)$ in $\mathbb{C}$. Find α^{-1} and $(\alpha^2 + \alpha + 1)^{-1}$ in $\mathbb{Q}(\alpha)$.

Problem 6 (Field extensions)

Let E be an extension of a field F. Suppose that E_1 and E_2 are subfields of E containing F. Prove that if $[E_1 : F]$ and $[E_2 : F]$ are primes and if $E_1 \neq E_2$, then $E_1 \cap E_2 = F$.

Problem 7 (Algebraic elements)

In (i) and (ii), show that the given number α is algebraic over $\mathbb{Q}$ by finding $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.

(i) $\alpha = 1 + i$.

(ii) $\alpha = \sqrt{1 + \sqrt[3]{2}}$.

Problem 8 (Minimal polynomials)

Find $[\mathbb{Q}(\sqrt{2} + i) : \mathbb{Q}]$.

Problem 9 (Algebraic extensions)

Let F be an extension of a field with q elements and let E be an extension of F. Suppose that $\alpha \in E$ is algebraic over F. Prove that $|F(\alpha)| = q^n$ for some positive integer n.

Problem 10 (Simple extensions)

Prove that $\mathbb{Q}(\sqrt{3} + \sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

Problem 11 (Splitting fields)

Find the splitting field for $x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$.

Problem 12 (Splitting fields)

Find the splitting field for $x^4 - x^2 - 2 \in \mathbb{Z}_3[x]$.