1 Contrasts

To illustrate the concepts in this chapter, we will use the example of the drink taste test discussed in earlier lectures. Suppose now that there are four groups: Soda Float, Calpis, Milk Tea and another green tea drink, with 3 subjects per group. As before, we will refer to the treatment groups as 1, 2, 3, and 4, and the response is a taste score. Refer to the SAS and R code for today's lecture. The term contrast is used to describe a comparison of means. Specifically, a contrast is a linear combination of the population means,

$$L = \sum_{i=1}^{g} w_i \mu_i \text{ that also satisfies } \sum_{i=1}^{g} w_i = 0.$$

We require that the coefficients w_i sum to zero so that the comparison is meaningful (we would not be interested in $\mu_1 - 3\mu_2$ for example). For contrasts we are generally interested in testing the null hypothesis $H_0 : L = \sum_{i=1}^{g} w_i \mu_i = 0$, against the alternative hypothesis $H_A : L = \sum_{i=1}^{g} w_i \mu_i \neq 0$. Notice that our text uses a slightly different notation to denote a contrast. Instead of the symbol L, which we use to express the value of the contrast, they use $w(\{\mu_i\})$, which focuses attention on the w_i values used to define the contrast. They also point out that the value of a contrast does not depend on the restrictions on the α_i values. As an example, if we wished to test whether the two non-tea drinks differed from each other, we can express the null hypothesis as $H_0 : 1\mu_1 - 1\mu_2 = \mu_1 - \mu_2 = 0$. Here $w_1 = 1$ and $w_2 = -1$ ($w_3 = w_4 = 0$), so $w_1 + w_2 + w_3 + w_4 = 0$ as required. This is an example of a **pairwise contrast**, which is defined as a contrast involving only two groups. An example of a **non-pairwise contrast** would be if we wished to test if the average taste of the two tea drinks differed from the average taste of the two non-tea drinks. We can express this null hypothesis as $H_0 : (\mu_1 + \mu_2)/2 - (\mu_3 + \mu_4)/2 = 0$. Here $w_1 = 1/2, w_2 = 1/2, w_3 = -1/2, w_4 = -1/2$, so again $w_1 + w_2 + w_3 + w_4 = 0$, as required by the definition of a contrast. If we wanted to test the Soda Float drink against the average of the two tea drinks, what would be the contrast? Another type of contrast is a polynomial contrast, which is a comparison among the levels of a quantitative factor (like a dose level) that correspond to a particular polynomial shape for the response. Contrasts for a linear trend or a quadratic trend are the two most commonly used polynomial contrasts.

A property of a set of contrasts, called orthogonality, is useful when considering sets of tests. Two contrasts

$$L_1 = \sum_{i=1}^{g} w_{1i} \mu_i \text{ and } L_2 = \sum_{i=1}^{g} w_{2i} \mu_i \text{ are orthogonal if } \sum_{i=1}^{g} \frac{w_{1i} w_{2i}}{n_i} = 0.$$

If the group sample sizes are equal then this is equivalent to $\sum_{i=1}^{g} w_{1i} w_{2i} = 0$. In the examples above, if we identify the first contrast as L_1 and the second contrast as L_2, then $w_{11} = 1, w_{12} = -1, w_{13} = w_{14} = 0$, are the coefficients for L_1 and $w_{21} = 1/2, w_{22} = 1/2, w_{23} = -1/2, w_{24} = -1/2$, are the coefficients for L_2. Then if the sample sizes are equal, $\sum_{i=1}^{g} w_{1i} w_{2i} = (1)(1/2) + (-1)(1/2) + (0)(-1/2) + (0)(-1/2) = 0$, so L_1 and L_2 are orthogonal. Orthogonal contrasts are statistically independent, so that the outcome of testing one contrast is independent of the outcome of testing the other contrast. In our example, whether or not the two non-tea drinks have different taste gives no information about whether the average of the non-tea drinks differ from the average of the two tea drinks. A set of more than two contrasts is mutually orthogonal if each pair of contrasts in the set is orthogonal to each other. The concept of a contrast or a set of contrasts at first seems somewhat esoteric, but in fact it is essential to understand these concepts to fully understand ANOVA, particularly in complicated situations. The contrasts that you will use should depend on the questions of scientific interest from your experiment.

2 Inference for Contrasts

We can estimate the contrast

$$L = \sum_{i=1}^{g} w_i \mu_i \text{ with } \widehat{L} = \sum_{i=1}^{g} w_i \overline{y}_{i\cdot}, \text{ and } \widehat{Var}(\widehat{L}) = MS_E \sum_{i=1}^{g} \frac{w_i^2}{n_i},$$

which leads to a t test of $H_0 : L = \sum_{i=1}^{g} w_i \mu_i = \delta$,

$$t = \frac{\widehat{L} - \delta}{s.e.(\widehat{L})} = \frac{\sum_{i=1}^{g} w_i \overline{y}_{i\cdot} - \delta}{\sqrt{\widehat{Var}(\widehat{L})}}.$$

For a two-tailed test, the t value is compared to $t_{\alpha/2,df}$, where *df* is the degrees of freedom for MSE (df = N-g for 1 way ANOVA). Confidence intervals for L can also be constructed as $\widehat{L} \pm t_{\alpha/2,df}\ s.e.(\widehat{L})$. Alternatively, we can compute a sum of squares for the contrast L:

$$SS_L = SS_w = \frac{(\sum_{i=1}^{g} w_i \overline{y}_{i\cdot})^2}{\sum_{i=1}^{g} \frac{w_i^2}{n_i}}.$$

3 Orthogonal contrasts form a partition of SS_{Trt}

As noted in the text, we can form as many orthogonal contrasts as we have degrees of freedom between groups. Essentially these orthogonal contrasts partition the SS_{Trt} into SS_{L_i} terms that allow us to separate the total between group sum of squares into parts attributable to different contrasts. This can be a powerful tool for understanding treatment effects. One example of this is when we use orthogonal polynomial contrasts to partition dose effects into parts attributable to linear trend, quadratic trend, and higher-order trends. For equally-spaced dosage levels with equal-sample-size groups, the coefficients for orthogonal polynomial contrasts are given in Table D.6 on page 630 of our text.