# Approximate F tests

In the factorial treatment design with one random factor and two fixed factors, we have for the random factor (called $C$ here)

$$E(MS_C) = \sigma^2 + n\sigma^2_{abc} + nb\sigma^2_{ac} + na\sigma^2_{bc} + nab\sigma^2_c$$

To test the null hypothesis $H_0 : \sigma^2_c = 0$, we would like to have a mean square with expectation $\sigma^2 + n\sigma^2_{abc} + nb\sigma^2_{ac} + na\sigma^2_{bc}$, but no single mean square satisfies this expression. However, we do have:

$$\begin{aligned} E(MS_{AC}) &= \sigma^2 + n\sigma^2_{abc} + nb\sigma^2_{ac} \text{ and} \\ E(MS_{BC}) &= \sigma^2 + n\sigma^2_{abc} + na\sigma^2_{bc} \end{aligned}$$

which sum to yield:

$$E(MS_{AC}) + E(MS_{BC}) = 2\sigma^2 + 2n\sigma^2_{abc} + nb\sigma^2_{ac} + na\sigma^2_{bc},$$

which has too many $\sigma^2 + n\sigma^2_{abc}$ terms. So if we substract the three-way interaction term we have

$$\begin{aligned} E(MS_{AC}) + E(MS_{BC}) - E(MS_{ABC}) &= 2\sigma^2 + 2n\sigma^2_{abc} + nb\sigma^2_{ac} + na\sigma^2_{bc} - (\sigma^2 + n\sigma^2_{abc}) \\ &= \sigma^2 + n\sigma^2_{abc} + nb\sigma^2_{ac} + na\sigma^2_{bc} \end{aligned}$$

Thus we can use the F statistic

$$F = \frac{MS_C}{MS_{AC} + MS_{BC} - MS_{ABC}}$$

to test $H_0 : \sigma^2_c = 0$. To approximate the distribution of this F statistic with an F distribution, we use Satterhwaite's approximation that a linear combination of mean squares, $M = \sum a_i MS_i$, has an approximate degrees of freedom of:

$$\nu = \frac{M^2}{\sum \frac{(a_i MS_i)^2}{\nu_i}}.$$

Satterhwaite's result is used to obtain the degrees of freedom for the denominator of the F statistic. The F statistic above is commonly used for this test, but the text points out that there are advantages to instead using the F statistic of:

$$F = \frac{MS_C + MS_{ABC}}{MS_{AC} + MS_{BC}}$$

so that negative values cannot occur.