1 Factorial Experiments, Treatment Structure, and Analysis

An experiment with more than one factor is a factorial experiment if all factor-level combinations are used. Two advantages of this approach are i) the ability to detect interactions between treatments, and ii) more efficiency (for all tests) than separate one-at-a-time experiments. Even if the data do not come from an experiment, but the grouping factor is arranged into all combinations of some other factors, then we can still say that the groups have factorial treatment structure. As an example, suppose that we measure blood pressure on six groups of people: Male children, Female children, Male young adults, Female young adults, Male middle aged adults, and Female middle ages adults. The factors of gender and age are not randomly applied, but we can talk about the six groups having factorial treatment structure in these two factors. Initially we will consider models for data where only two factors are used, and also we will at first only consider balanced data, in which all treatment combinations have the same number, n, of replicates.

1.1 Data with factorial treatment structure

Suppose that we have an experiment with two factors, one at three levels, and the other at two levels. We can denote a response by y_{ijk}, where $i = 1, ..., a$, and $j = 1, ..., b$, index the two treatment factors and $k = 1, ..., n$ indexes the replicates within a treatment combination. In the situation mentioned above, $a = 3$, $b = 2$, and suppose there are 5 replicates per group so that $n = 5$.Then there are a total of 30 observations, and if we ignore the factorial nature of the experiment and just consider the $3 * 2 = 6$ different groups, the 'skeleton' ANOVA table would be:

Source	degrees of freedom
Combinations	
Error	
Total	

However, we also know that a test for differences among the first factor levels has $a = 1 = 3 - 1 = 2$ df, and that a test for differences among the second factor levels has $b - 1 = 2 - 1 = 1$ df, so that the ANOVA table could look like:

Source	degrees of freedom
Factor A	
Factor B	
?	
Error	
Total	

The tests for Factor A and Factor B are called 'main effect tests'. But we are missing something in the table. It is the other source of variation among the $a * b$ combination means called the interaction effect. It is easy to see the interpretation of these tests by viewing a profile plot, which is a plot of the combination means. Usually we observe the profile plot of the sample means $\overline{y}_{ij\cdot}$, but we can also look at population means μ_{ij} to understand the idea of these tests. Refer to the figure with this lecture to see the a comparison of main effects versus interaction effects with factorial experiments. So the skeleton ANOVA table for a factorial analysis is then:

Source	degrees of freedom
Factor A	
Factor B	
Interaction	
Error	
Total	

See the table on page 181 of the text to show the full layout of the ANOVA table for analysis of a two-factor factorial design.

1.2 Models for data with factorial treatment structure

As with the simpler Completely Randomized Design, with a Completely Randomized Factorial (CRF) design we can use a means model:

$$y_{ijk} = \mu_{ij} + \varepsilon_{ijk},$$

Or an effects model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk}.$$

To connect the models we note that

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij}$$

1.3 Contrasts for factorial effects

It is typically easier to specify contrasts using the means model form. Here is an example for a **main effects contrast** for the A factor:

$$l_{A1} = \mu_{1\cdot} - \mu_{2\cdot},$$

which is comparing level 1 of the A factor to level 2, averaging over the B factor (that is the dot notation). Here is an example of a contrast for the B factor:

$$l_{B1} = \mu_{\cdot 1} - \mu_{\cdot 2},$$

which compares the two levels of the B factor averaging over the A factor. Here is an example of an **interaction contrast**:

$$l_{AB1} = (\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22}),$$

which is checking to see if the difference between levels 1 and 2 of factor B differ between levels 1 and 2 of the A factor. Contrasts that compare levels of one factor while holding all other factors at a common level are called **simple effect contrasts**. For example, $\mu_{11} - \mu_{12}$ is a simple effect. One way to understand interaction contrasts is that they examine differences in simple effects.