Assessing Students' Levels of Understanding Multiplication through

Problem Writing

Assessing student learning is a very important part of teaching. As teachers, we are continually searching for assessments that give us valid information about what our students are learning. Occasionally, the assessments we use can surprise us with the results they yield. Jillian, a third-grade teacher, had such an experience when she asked her students to solve the following problem:

> There are three boxes of chicken nuggets on the table. Each box contains six chicken nuggets. How many chicken nuggets are there in all?

After some time, Jillian asked Johnny to explain to the class how he solved the problem. Knowing that Johnny had learned all of his multiplication facts, she felt that Johnny understood multiplication and was hopeful that he could help other students understand how to solve multiplication word problems. Instead, Johnny reported that he added to get his answer of nine. Surprised by his response, Jillian asked him to explain how he got his answer. Johnny replied, "Well, the question says 'how many

in all,' which means add. Three plus six is nine."

Have you ever found yourself in this situation? You think your students really understand something because they have performed well on your assessments only to find out later that their understanding is incomplete. Typically, we look back at the information provided by our assessments, and we know what students can or cannot do. As in the previous scenario, the teacher knew that Johnny could provide the answers to multiplication facts. However, many times our assessments fall short of completing the picture of what students know and understand. Identifying an assessment tool that can help complete this picture is essential.

For the past three years, we worked with teachers facing similar assessment issues in four different school districts. One of the strategies introduced to teachers was problem writing, which engages students in creating mathematical word problems based on a given prompt that can be designed to match the mathematics being studied (see **fig. 1**). Barlow and Cates (2006–2007) describe problem writing as a worthwhile mathematical task that positively influences students' mathematical understandings, problem solving skills, and mathematical dispositions. For these reasons alone, we believe problem writing should be incorporated into any elementary classroom. Within our school districts, though, as students began writing problems, teachers began recognizing the invaluable assessment information contained in these problems. Therefore, the purpose of this article is to demonstrate

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inability to formulate the mathematics in the word problem provides information regarding the level of mathematical understanding.

Second, is the problem's question appropriate? As we read through student problems, we noticed that students were not always able to formulate a correct question. This was particularly interesting when students were able to represent the mathematics correctly but then asked an incorrect question, thus indicating a different level of understanding, worthy of review. For example, if writing a word problem for $24 \div 3$, a student might begin, "Mary has twenty-four cupcakes. She wants to give each friend three cupcakes." This is a mathematically correct scenario. There are twenty-four cupcakes, and they are being divided into groups of three. However, the student's question might read, "How many cupcakes will each friend get?" The answer to this question is three, and the problem does not match the expression $24 \div 3$ described in the prompt.

To provide an example of using problem writing as an assessment tool, we asked forty-five sixth-grade students from two suburban middle schools to write a series of word problems that included one that could be represented by the expression 4×8. All of the sixth graders were either on grade level or above. Before we consider the students' work, we need to give attention to the mathematics in the expression 4×8, namely multiplication.

The General Model of Multiplication

The general model of multiplication states that $n \times a$ means n groups of a, or the amount a added n times. If $n \times a = b$, then n is called the *multiplier*, a is called the *multiplicand*, and b is called the *product* (Bassarear 2005). In the case of 4×8, 4 is the multiplier, 8 is the multiplicand, and 4×8 represents 4 groups of 8, or $8 + 8 + 8 + 8$. Note that 4×8 and 8×4 are equivalent expressions in value but are different in representation. That is to say, 4×8, or 4 groups of 8, is not the same as 8×4, or 8 groups of 4. Students who recognize that 4×8 is modeled differently from 8×4 have a deeper understanding of multiplication than do those students who fail to recognize this difference; the former are aware of the role of the multiplier and the multiplicand.

In addition to understanding the

the power of problem writing in assessing students' mathematical understandings.

Indicators of Understanding

In using problem writing as an assessment tool, one must first decide what to look for in the problems. What will provide insight into students' mathematical understandings? What are the indicators that show understanding or lack of understanding? After examining numerous student-written problems, we identified two questions to ask when using problem writing as an assessment tool.

First and foremost, does the mathematics contained in the problem correctly represent the mathematics called for in the prompt? For example, if the students were asked to create a word problem that could be represented by $24 \div 3$, did they write a division problem? Did they divide twenty-four by three, or did they become confused and divide twenty-four by eight? Clearly, a student's ability or

Figure 1

Sample Prompts

- The answer is 32 cents. Create the word problem.
- Create a word problem that involves subtraction and division.
- Write a word problem that involves averaging.
- Examine the graph provided. Write at least four different word problems that can be answered using the graph.

general model of multiplication, we should also be aware that four classes of multiplication problems exist (Greer 1992). The general model accounts for two of these problem types, namely repeated addition problems and rate problems. The remaining two problem types involve the area model and the Cartesian product model (Van de Walle 2007). The last two models will not be discussed in this context because none of the sixth-grade students wrote problems involving either of these models.

Examining Students' Levels of Understanding Multiplication

In an effort to uncover students' understandings of multiplication, students were asked to write a word problem that could be represented by the expression 4×8. Solving the expression 4×8 is a low-level task in terms of cognitive demand for sixth graders, because they have been exposed to the concept for several years. However, the task of creating a word problem is a higher-level task and, as such, can reveal much more about students' understandings than asking for the product of a basic fact (Smith and Stein 1998). The decision to have sixth-grade students write a multiplication problem was based on two major considerations. First, doing so provided an avenue for teachers to examine the students' depth of understanding of a concept that most teachers—using traditional assessments—would have assumed students understood well. Second, understanding multiplication is a critical step in students' development of understanding concepts that are typically taught in the sixth grade, such as multiplication of fractions.

For each of the six student-written problems that follow, the students' understandings and misunderstandings as revealed through the problem-writing process will be identified. Additionally, suggestions for addressing the misunderstandings will be provided.

Tires and rims

In the Tires and Rims problem (see **fig. 2**), the student demonstrated that he knew that 4×8 is equal to 32, as indicated in his drawing. However, there was no evidence in the word problem or in the drawing that he understood multiplication as repeated addition or as groups of. One might infer that the student has memorized his multiplication facts but does not understand what these facts represent.

In our sample of sixth graders, 11 percent of the

students wrote problems similar to the Tires and Rims problem. In each of these problems, the students used the numbers 4, 8, and/or 32 but did not correctly represent the multiplication fact in any way. We found this to be a surprising result, considering the simplicity of the problem. When working with these students, it is important to interview the student to determine whether the student's understanding of multiplication is limited to memorized facts. Alternatively, the student may understand multiplication but cannot demonstrate this understanding through the creation of a word problem. In either case, after the interview, the teacher should guide the student to represent 4×8 with pictures. The student should then be guided to use the pictures to create an appropriate word problem. Finally, the student should write the accompanying number sentence.

Puppies

Figure 3 contains the Puppies problem. This student included the product of thirty-two in the problem, indicating her knowledge of the multiplication fact $4 \times 8 = 32$. From the student's drawing, one can also see the four groups of eight that produced the answer of thirty-two. However, the word problem represents division ($32 \div 4$) rather than multiplication.

Interestingly, 7 percent of the sample provided problems that represented division. While the mathematics contained in the problem are correct, they do not match the mathematics called for in the prompt. In this situation, the teacher should ask these students to explain the origin of the thirty-two in the problem. Based on student responses, it may be that the teacher needs to work with the students to differentiate between the meaning of division and the meaning of multiplication. Alternatively, the difficulty may lie in the fact that the student did not realize that the answer to the word problem should be thirty-two; that is to say, she did not understand the task at hand. If this is the case, the student should be provided with the opportunity to write another word problem, following a clarification of the task requirements.

Fish tanks

In the Fish Tanks problem (see **fig. 4**) the student has demonstrated that he understands multiplication as either groups of or repeated addition. The multiplication scenario, however, describes 8 groups of 4 rather than 4 groups of 8. This indicates that the student does not recognize the role of the multiplicand and the multiplier, which would require a deeper understanding of multiplication. Similarly, 31 percent of the sixth graders in this sample successfully created

a multiplication scenario yet failed to correctly represent 4 as the multiplier and 8 as the multiplicand. Most likely, these students believe that representations for 4×8 are the same as that for 8×4 because they yield the same answer. This level of understanding of multiplication is rarely, if ever, measured on a typical assessment. In attempting to teach a rich mathematics curriculum, measuring deeper levels of understanding is imperative.

In this situation, teachers should provide students who have written problems similar to the Fish Tanks problem the opportunity to compare and contrast their problems with someone's problem that accurately represents the expression 4×8. They could also discuss whether 4 groups of 8 is equivalent to 8 groups of 4. After being faced with this conflict, the teacher could help students resolve the conflict by explaining the role of the multiplicand and the multiplier and then having them write new problems for 4×8 or some other expression involving multiplication.

Rescue workers

Take a moment to answer the question in the Rescue Workers problem (see **fig. 5**). While the multiplication scenario for the problem is correct, the question does not yield the answer thirty-two. The student asks, "How many people were in each group?" The answer to that question is eight. An appropriate question could have been, "How many people were there in all?" In our sample, 13 percent of the students created problems similar to this one by accurately representing a multiplication scenario but failing to ask an appropriate question. One explanation is that this error represents a misunderstanding of what the product symbolizes. Alternatively, it could represent carelessness on the part of the student. The source of the error can be clarified by asking the student to solve his problem. If the error is the result of carelessness on the part of the student, then the student will most likely self-correct. If the student does not readily self-correct, then he should be asked to explain why he believes his problem correctly illustrates 4×8, as well as what the product should symbolize. One approach the teacher could take to develop the student's understanding of how the product is presented in a word problem would be to begin by having the student state the product of 4×8, namely 32. Then, the student could be guided to identify where the product 32 can be derived from his problem scenario, which in this case would be the total number of rescue workers. Finally, the teacher could discuss with the student how this information can be used to frame an appropriate question.

Figure 2

Tires and Rims problem

Jimmy needed help to solve this math problem:

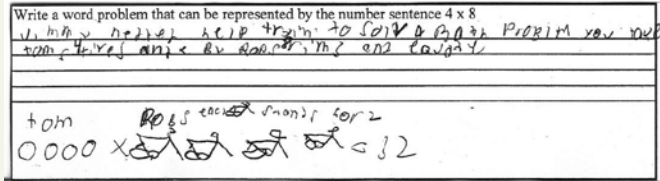You have Tom's 4 tires, and you buy Rob's 8 rims and [it] equals [what?]

Figure 3

Puppies problem

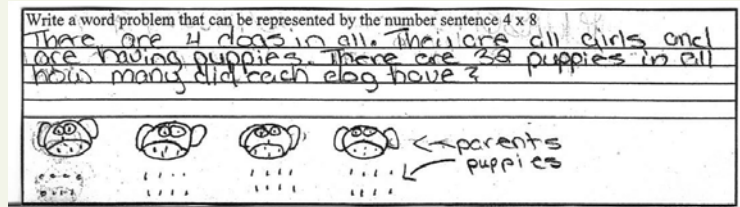There are 4 dogs, and all of them are having puppies. There are 32 puppies in all. How many puppies did each dog have?

Figure 4

Fish Tanks problem

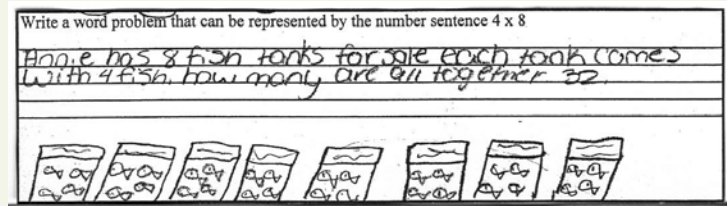Amy has 8 fish tanks for sale. Each tank comes with 4 fish. How many all together? (thirty-two)

Figure 5

Rescue Workers problem

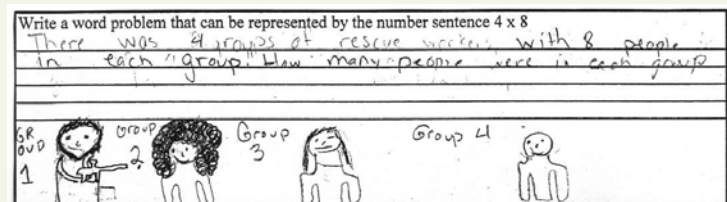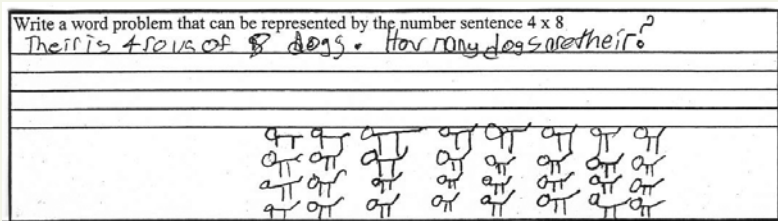There were 4 groups of rescue workers with 8 people in each group. How many people were in each group?
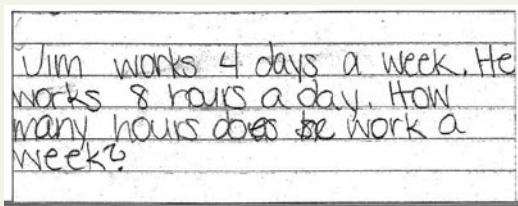
(a)
There are 4 rows of 8 dogs. How many dogs are there?

(b)
Jim works 4 days a week. He works 8 hours a day. How many hours does he work a week?

Rows of dogs versus work

Consider the problems contained in **figure 6**. Both of these students have correctly represented 4×8 and have included a question that calls for the product 32. Both have likely written about ideas tied to their own worlds. What is the difference between these two problems? In the Rows of Dogs problem, is the idea of 32 dogs lining up in rows realistic? Now read the Work problem. Does this seem realistic? Calculating the number of hours worked in a week depicts a realistic problem that occurs outside the classroom. The student who wrote the Rows of Dogs problem chose not to write a realistic problem but may have been able to do so if asked. However, is it important to expect students to write realistic problems? We believe it is. Students should be asked to create problems that emerge from realistic situations because doing so equips them to transfer the knowledge being gained in the classroom to problems that arise outside the classroom. One way to facilitate students' ability to create realistic problems is through comparing realistic and unrealistic problems written by other students. Attention should be given to what constitutes a realistic problem and why making realistic connections is important.

Insight gained from problems

The depth of students' understanding of multiplication was readily identified through these examples. The basic fact 4×8 would likely have been the only level of understanding addressed on a typical multiplication assessment. Almost all students in this sample provided evidence of their knowledge of the multiplication fact $4 \times 8 = 32$, and therefore most of us would have assumed that the students had a full understanding of 4 times 8. This problem-writing assessment, however, exposed a variety of levels of understanding and misunderstanding multiplication. Note that even if a typical multiplication assessment includes word problems, most students will readily multiply the numbers contained in the problem because they know that they are taking a multiplication test.

Conclusion

Using problem writing as an assessment can reveal students' understandings and misunderstandings in a manner in which traditional assessments cannot. As seen in the examples provided, gaps in students' understanding of multiplication, the role of the multiplicand and the multiplier, and the meaning of the product were readily assessed via problem writing. The assessment information gleaned from the students' word problems could be used by the teacher in several ways. First, this information could be used to tailor classroom tasks to meet the different needs of the students. Second, students could be grouped, either homogenously or heterogeneously. Third, problem writing could be used as a diagnostic tool. The information could be used to guide instructional decisions if the concept assessed in the problem writing is contained in upcoming chapters. Fourth, this assessment tool could assist in developing specific remediation plans by identifying gaps in students' understandings.

An additional reason one might want to utilize problem writing as an assessment tool is that it holds the potential for fully engaging students in several of the National Council of Teachers of Mathematics' (NCTM) Process Standards (2000). As students write and discuss their problems, they are revealing their mathematical thinking, which supports the Communication Standard. Class discussions that call on students to explain their thinking or justify the accuracy of the mathematical scenarios they create in their problem writing engage students in experiences that are in line with the expectations of the Reasoning and Proof Standard. Additionally, requiring students to create realistic problems and illustrate these problems with drawings aligns with the expectations of the Connections and Representation Standards, respectively. With regard to the Problem Solving Standard, students who are writing word problems, although not actually engaged in the process of solving problems, are enhancing their problem-solving skills through thinking creatively about word problems and through

reasoning about the structure necessary to write word problems correctly.

When problem writing is used as an assessment tool, the teacher must probe to find the sources of student errors and determine whether errors are due to lack of understanding, misconceptions, carelessness, poorly developed writing skills, limited language, lack of instruction on how to write a story problem, or some other difficulty. One reason that identifying the source of error is important is that writing a word problem is an acquired skill. Students may need to be taught the elements of effective problem writing in the same manner that they are taught to write a summary or a narrative. As seen in the student problems provided, students' writing skills vary greatly. However, note that these problems are first drafts that have not gone through an editing process. As such, the samples may not be indicative of what students can or should be able to do in terms of their actual writing skills. They do, however, reflect the information that can be gathered through the use of this assessment tool in terms of mathematical understanding.

In this day of standards-based instruction, we expect students to develop a deeper understanding of the mathematics they are studying. Problem writing provides teachers with a tool for assessing this type of knowledge. By incorporating problem writing into your existing assessments, the benefits for you and your students are infinite.

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