It was pointed out in the introduction to Part II that potential flows may be analyzed in a much simpler way than general fluid flows. Within the category of potential flows, the two-dimensional subset lends itself to even greater simplification. It will be shown in this chapter that the simplification is so great that solutions to Eqs. (1.1.5) and (1.1.6) may be obtained without actually solving any differential equations. This is achieved through use of the powerful tool of complex variable theory.

The chapter begins by introducing the stream function, which together with the velocity potential, leads to the definition of a complex potential. Through this complex potential, some elementary solutions corresponding to sources, sinks, and vortices are examined. The superposition of such elementary solutions then leads to the solution for the flow around a circular cylinder. The method of conformal transformations is then introduced, and the Joukowski transformation is used to establish the solutions for the flow around ellipses and airfoils. The Schwarz-Christoffel transformation is then introduced and used to study the flow in regions involving sharp corners. Included in this chapter are examples of free-surface configurations.

4.1 STREAM FUNCTION

The velocity potential \( \phi \) was defined in such a way that it automatically satisfied the condition of irrotationality. The continuity equation then showed that \( \phi \) had to be a solution of Laplace's equation. A second function may be defined by a
complementary procedure for two-dimensional incompressible fluid flows. That is, a function may be defined in such a way that it automatically satisfies the continuity equation, and the equation which it must satisfy will be determined by the condition of irrotationality.

The continuity equation, in cartesian coordinates, for the flow field under consideration is

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

Now introduce a function \( \psi \) which is defined as follows:

\[ u = \frac{\partial \psi}{\partial y} \quad \text{(4.1a)} \]
\[ v = -\frac{\partial \psi}{\partial x} \quad \text{(4.1b)} \]

With this definition, the continuity equation is satisfied identically for all functions \( \psi \). The function \( \psi \) is called the stream function, and by virtue of its definition it is valid for all two-dimensional flows, both rotational and irrotational.

The equation which the stream function \( \psi \) must satisfy is obtained from the condition of irrotationality. Denoting the components of the vorticity vector \( \omega \) by \( \xi, \eta, \zeta \), it is first observed that, in two dimensions, the only nonzero component of the vorticity vector is \( \zeta \), the component which is perpendicular to the plane of the flow. Secondly, it is noted that \( \zeta = \partial v/\partial x - \partial u/\partial y \). Thus, the condition of irrotationality is

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \]

Substituting for \( u \) and \( v \) from Eqs. (4.1) shows that \( \psi \) must satisfy the following equation:

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{(4.2)} \]

That is, the stream function \( \psi \), like the velocity potential \( \phi \), must satisfy Laplace's equation. The stream function \( \psi \) has some useful properties which will now be derived.

The flow lines which correspond to \( \psi = \text{constant} \) are the streamlines of the flow field. To show this, it is noted that \( \psi \) is a function of both \( x \) and \( y \) in general so that the total variation in \( \psi \) associated with a change in \( x \) and a change in \( y \) may be calculated from the expression

\[ d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v \, dx + u \, dy \]

where Eqs. (4.1) have been used. Then the equation of the line \( \psi = \text{constant} \) will be

\[ 0 = -v \, dx + u \, dy \]

or

\[ \left( \frac{dy}{dx} \right)_{\psi} = \frac{v}{u} \]

where the subscript denotes that this expression for \( dy/dx \) is valid for \( \psi \) held constant. But it was shown in Chap. 2 that this is precisely the equation of the streamlines in the \( xy \) plane, hence the lines corresponding to \( \psi = \text{constant} \) are the streamlines, and each value of the constant defines a different streamline. It is this property of the function \( \psi \) which justifies the name stream function.

Another property of the stream function \( \psi \) is that the difference of its values between two streamlines gives the volume of fluid which is flowing between these two streamlines. To show this, consider two streamlines corresponding to \( \psi = \psi_1 \) and \( \psi = \psi_2 \) as shown in Fig. 4.1. A control surface \( AB \) of arbitrary shape but positive slope is shown joining these two streamlines, and an element of this surface shows the positive volumetric flow rates crossing it in the \( x \) and \( y \) directions per unit depth perpendicular to the flow field. Then the total volume of fluid flowing between the streamlines per unit time per unit depth of flow field will be

\[ Q = \int_A^B u \, dy - \int_A^B v \, dx \]

But it was observed earlier that \( d\psi = -v \, dx + u \, dy \), so that, integrating this expression between the two points \( A \) and \( B \), it follows that

\[ \psi_2 - \psi_1 = -\int_A^B v \, dx + \int_A^B u \, dy \]

Comparing these two expressions confirms that \( \psi_2 - \psi_1 = Q \).
Finally, it should be noted that the streamlines \( \psi \) constant and the lines \( \phi \) constant, which are called *equipotential lines*, are orthogonal to each other. This may be shown by noting that if \( \phi \) depends upon both \( x \) and \( y \), the total change in \( \phi \) associated with changes in both \( x \) and \( y \) will be

\[
\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = -u \, dx + \nu \, dy
\]

where Eq. (II.4) has been used. Then the lines corresponding to \( \phi = \) constant will be defined by

\[
\frac{dy}{dx} = \frac{u}{\nu}
\]

That is,

\[
\frac{dy}{dx}_\phi = -\frac{1}{\frac{dy}{dx}}
\]

In words, the slope of the lines \( \phi = \) constant is the negative reciprocal of the slope of the lines \( \psi = \) constant, so that these sets of lines must be orthogonal. This property of the streamlines and the equipotential lines is the basis of a numerical procedure for solving two-dimensional potential-flow problems. The method is referred to as the *flow net*.

### 4.2 COMPLEX POTENTIAL AND COMPLEX VELOCITY

The velocity components \( u \) and \( \nu \) may be expressed in terms of either the velocity potential or the stream function. From Eqs. (II.4) and (4.1), these expressions are

\[
\begin{aligned}
  u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\
  \nu &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}
\end{aligned}
\]

That is, the functions \( \phi \) and \( \psi \) are related by the expressions

\[
\begin{aligned}
  \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\
  \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}
\end{aligned}
\]

But these will be recognized as the Cauchy-Riemann equations for the functions \( \phi(x, y) \) and \( \psi(x, y) \). Then consider the *complex potential* \( F(z) \), which is defined as follows:

\[
F(z) = \phi(x, y) + i\psi(x, y)
\]

where \( z = x + iy \). Now if \( F(z) \) is an analytic function, it follows that \( \phi \) and \( \psi \) will automatically satisfy the Cauchy-Riemann equations. That is, for every analytic function \( F(z) \) the real part is automatically a valid velocity potential and the imaginary part is a valid stream function.

The foregoing result suggests a very simple way of establishing solutions to the equations of two-dimensional potential flows. By equating the real part of a given analytic function to \( \phi \) and the imaginary part to \( \psi \), the theory of complex variables guarantees that \( \nabla^2 \phi = 0 \) and \( \nabla^2 \psi = 0 \) as required. The flow field corresponding to that analytic function may be determined by studying the streamlines \( \psi = \) constant. The corresponding velocity components may be calculated from Eqs. (II.4) or (4.1), and the pressure may be obtained using Eq. (II.6). This approach has the disadvantage of being inverse in the sense that a problem is first solved and then examined to see what the physical problem was in the first place. However, for teaching purposes this is of no consequence. Another disadvantage is that the method cannot be generalized to three-dimensional potential flows. On the other hand, this approach avails itself of the powerful results of complex variable theory and avoids the difficulties of solving partial differential equations. For these reasons the complex-potential approach will be used in this chapter.

Another quantity of prime interest, apart from the complex potential \( F(z) \), is the derivative of \( F(z) \) with respect to \( z \). Since \( F(z) \) is supposed to be analytic, \( dF/dz \) will be a point function whose value is independent of the direction in which it is calculated. Then, denoting this derivative by \( W \), its value will be given by

\[
W(z) = \frac{dF}{dz} = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}
\]

that is,

\[
W(z) = dF = u - i\nu
\]

where use has been made of Eqs. (4.3), (II.4), and (4.1b). In view of this result the quantity \( W(z) \) is called the *complex velocity*, although its imaginary part is \( -\nu \). Equation (4.4) offers a convenient alternative to Eqs. (II.4) and (4.1) for finding the velocity components corresponding to a given complex potential.

A useful property of the complex velocity is that, when multiplied by its own complex conjugate, it gives the scalar product of the velocity vector with itself. To show this, consider \( W(z) \) and its complex conjugate \( \overline{W}(z) \). Then

\[
W\overline{W} = (u - i\nu)(u + i\nu) = u^2 + \nu^2
\]

as required.
The significance of this result is that the quantity \( u \cdot u = \nabla \phi \cdot \nabla \phi = u^2 + v^2 \) appears in the Bernoulli equation.

Frequently it is advantageous to work in cylindrical coordinates rather than cartesian coordinates. An expression for the complex velocity may be readily obtained in cylindrical coordinates by converting the cartesian components of the velocity vector \((u, v)\) to cylindrical components \((u_\theta, u_\rho)\). Figure 4.2 shows a velocity vector \(OP\) decomposed into its cartesian components (shown solid) and also its cylindrical components (shown dotted). From this figure each of the cartesian velocity components may be expressed in terms of the two cylindrical components as follows:

\[
\begin{align*}
  u &= u_\rho \cos \theta + u_\theta \cos \left( \frac{\pi}{2} - \theta \right) = u_\rho \cos \theta - u_\theta \sin \theta \\
  v &= u_\rho \sin \theta + u_\theta \sin \left( \frac{\pi}{2} - \theta \right) = u_\rho \sin \theta + u_\theta \cos \theta
\end{align*}
\]

Substituting these expressions into Eq. (4.4) gives the expression for the complex velocity \(W\) in terms of \(u_\rho\) and \(u_\theta\):

\[
W = (u_\rho \cos \theta - u_\theta \sin \theta) - i(u_\rho \sin \theta + u_\theta \cos \theta)
\]

\[
= u_\rho (\cos \theta - i \sin \theta) - i u_\theta (\cos \theta - i \sin \theta)
\]

that is,

\[
W = (u_\rho - i u_\theta)e^{-i\theta} \quad (4.6)
\]

The foregoing results [Eqs. (4.3) to (4.6)] are sufficient to establish the flow fields, which are represented by simple analytic functions.

### 4.3 UNIFORM FLOWS

The simplest analytic function of \(z\) is proportional to \(z\) itself, and the corresponding flow fields are uniform flows.

**Figure 4.2**
Decomposition of a velocity vector \(OP\) into its cartesian components \((u, v)\) and its cylindrical components \((u_\rho, u_\theta)\).

**Figure 4.3**
Uniform flow in (a) the \(x\) direction, (b) the \(y\) direction, and (c) an angle \(\alpha\) to the \(x\) direction.

First, consider \(F(z)\) to be proportional to \(z\) where the constant of proportionality is real. That is,

\[
F(z) = cz
\]

where \(c\) is real. Then, from Eq. (4.4),

\[
W(z) = u - i v = c
\]

Then, by equating real and imaginary parts of this equation, the velocity components corresponding to this complex potential are

\[
\begin{align*}
  u &= c \\
  v &= 0
\end{align*}
\]

But this is just the velocity field for a uniform rectilinear flow as shown in Fig. 4.3a. Thus the complex potential for such a flow whose velocity magnitude is \(U\) in the positive \(x\) direction will be

\[
F(z) = Uz \quad (4.7a)
\]

Next consider the complex potential to be proportional to \(z\) with an imaginary constant of proportionality. Then

\[
F(z) = -icz
\]

where \(c\) is real. The minus sign has been included to make the velocity component positive when \(c\) is positive. For this complex potential

\[
W(z) = u - i v = -ic
\]

so that the velocity components are

\[
\begin{align*}
  u &= 0 \\
  v &= c
\end{align*}
\]

This is a uniform vertical flow as shown in Fig. 4.3b. Then the complex potential for such a flow whose velocity magnitude is \(V\) in the positive \(y\) direction will be

\[
F(z) = -iVz \quad (4.7b)
\]
Finally, consider a complex constant of proportionality so that
\[ F(z) = ce^{-iaz} \]
where \( c \) and \( a \) are real. For this complex potential
\[ W(z) = u - iv = c \cos \alpha - ic \sin \alpha \]
Hence the velocity components of the flow field are
\[ u = c \cos \alpha \]
\[ v = c \sin \alpha \]
This corresponds to a uniform flow inclined at an angle \( \alpha \) to the \( x \) axis as shown in Fig. 4.3c. Hence the complex potential for such a flow whose velocity magnitude is \( V \) will be
\[ F(z) = Ve^{-iaz} \quad (4.7c) \]
This last result, of course, contains the two previous results as special cases corresponding to \( \alpha = 0 \) and \( \alpha = \pi/2 \).

4.4 SOURCE, SINK, AND VORTEX FLOWS

Complex potentials which correspond to the flow fields generated by sources, sinks, and vortices are obtained by considering \( F(z) \) to be proportional to \( \log z \). When considering \( \log z \), we consider the principal part of this multivalued function corresponding to \( 0 < \theta < 2\pi \).

Consider, first, the constant of proportionality to be real. Then
\[ F(z) = c \log z \]
\[ = c \log Re^{i\theta} \]
\[ = c \log R + ic\theta \]
Hence, from Eq. (4.3),
\[ \phi = -c \log R \]
\[ \psi = -c\theta \]
That is, the equipotential lines are the circles \( R = \text{constant} \) and the streamlines are the radial lines \( \theta = \text{constant} \). This gives a flow field as shown in Fig. 4.4a in which the streamlines are shown solid and the direction of the flow is shown for \( c > 0 \). The direction of the flow is readily confirmed by evaluating the velocity components. In view of the geometry of the flow, cylindrical coordinates are preferred, so that
\[ W(z) = \frac{c}{z} = \frac{c}{R} e^{-ia\theta} \]

Comparison with Eq. (4.6) shows that the velocity components are
\[ u_R = \frac{c}{R} \]
\[ u_\theta = 0 \]
which confirms the directions indicated in Fig. 4.4a for \( c > 0 \).

The flow field indicated in Fig. 4.4a is called a source. The velocity is purely radial and its magnitude decreases as the flow leaves the origin. In fact, the origin is a singular point corresponding to infinite velocity, and as the fluid flows radially outwards, its velocity is decreased in such a way that the volume of fluid crossing each circle is constant, as required by the continuity equation.

Sources are characterized by their strength, denoted by \( m \), which is defined as the volume of fluid leaving the source per unit time per unit depth of the flow field. From this definition it follows that
\[ m = \int_0^{2\pi} u_R R d\theta \]
\[ = \int_0^{2\pi} c R d\theta = 2\pi c \]
Here, the result \( u_R = c/R \) has been used. Then \( c \) may be replaced by \( m/2\pi \), giving the following complex potential for a source of strength \( m \):
\[ F(z) = \frac{m}{2\pi} \log z \]
The source corresponding to this complex potential is located at the origin, the location of the singularity. Then the complex potential for a source of strength
m located at the point \( z = z_0 \) will be
\[
F(z) = \frac{m}{2\pi} \log(z - z_0) \quad (4.8)
\]
Clearly, the complex potential for a sink, which is a negative source, is obtained by replacing \( m \) by \(-m\) in Eq. (4.8).

Now consider the constant of proportionality in the logarithmic complex potential to be imaginary. That is, consider
\[
F(z) = -ic \log z
\]
where \( c \) is real and the minus is included to give a positive vortex. Then, using cylindrical coordinates,
\[
F(z) = -ic \log R e^{i\theta} = -ic \log R
\]
Then, from Eq. (4.3), the velocity potential and the stream function are
\[
\phi = c\theta \\
\psi = -c \log R
\]
That is, the equipotential lines are the radial lines \( \theta = \) constant and the streamlines are the circles \( R = \) constant as shown in Fig. 4.4b. The velocity components may be evaluated by use of the complex velocity.
\[
W(z) = -i \frac{c}{z} = -i \frac{c}{R} e^{-i\theta}
\]
Comparison with Eq. (4.6) shows that the velocity components are
\[
u_r = 0 \\
u_\theta = \frac{c}{R}
\]
Hence the direction of the flow is positive (counterclockwise) for \( c > 0 \), and the resulting flow field is called a vortex.

A vortex is characterized by its strength, which may be measured by the circulation \( \Gamma \) which is associated with it. From Eq. (2.3), the circulation \( \Gamma \) which is associated with the singularity at the origin is
\[
\Gamma = \oint u \cdot dl = \int_0^{2\pi} u_\theta R \, d\theta = -c \theta - 2\pi c
\]
Here, the result \( u_\theta = c/R \) has been used. Then \( c \) may be replaced by \( \Gamma/2\pi \), giving the following complex potential for a positive (counterclockwise) vortex of strength \( \Gamma \):
\[
F(z) = -i \frac{\Gamma}{2\pi} \log z
\]
The singularity in this expression is located at \( z = 0 \). That is, the line vortex is located at \( z = 0 \). Then the complex potential for a positive vortex located at \( z = z_0 \) will be
\[
F(z) = -i \frac{\Gamma}{2\pi} \log(z - z_0) \quad (4.9)
\]
The complex potential for a negative vortex would be obtained by replacing \( \Gamma \) by \(-\Gamma\) in Eq. (4.9). Note, however, that the negative coefficient is associated with the positive vortex.

The flow field represented by Eq. (4.9), which is shown in Fig. 4.4b for \( z_0 = 0 \), corresponds to a so-called free vortex. That is, for any closed contour which does not include the singularity, the circulation will be zero and the flow will be irrotational. All circulation and vorticity associated with this type of vortex is concentrated at the singularity. This is in contrast with the solid-body rotation vortex which was mentioned in Chap. 2.

The principal application of the source, the sink, and the vortex is in the superposition with other flows to yield more practical flow fields.

4.5 FLOW IN A SECTOR
The flows in sharp bends or sectors are represented by complex potentials which are proportional to \( z^n \), where \( n \geq 1 \). A special case of such complex potentials would be \( n = 1 \), which represents a uniform rectilinear flow. Then, in order that this special case would reduce to Eq. (4.7a), consider the complex potentials
\[
F(z) = U z^n
\]
Substituting \( z = Re^{i\theta} \) and separating the real and imaginary parts of this function gives
\[
F(z) = UR^n \cos n\theta + iUR^n \sin n\theta
\]
Then the velocity potential and the stream function are
\[
\phi = UR^n \cos n\theta \\
\psi = UR^n \sin n\theta
\]
From this it is evident that when \( \theta = 0 \) and when \( \theta = \pi/n \), the stream function \( \psi \) is zero. That is, the streamline \( \psi = 0 \) corresponds to the radial lines \( \theta = 0 \) and
\( \theta = \pi/n \). Between these two lines, the streamlines are defined by \( R^n \sin n\theta = \) constant. This gives the flow field shown in Fig. 4.5. The direction of the flow along the streamlines may be determined from the complex velocity as follows:

\[
W(z) = nUz^{n-1} = nUR^{n-1}e^{i(n-1)\theta} = (nUR^{n-1}\cos n\theta + inUR^{n-1}\sin n\theta)e^{-i\theta}
\]

Thus, by comparison with Eq. (4.6), the velocity components are

\[
u_{r} = nUR^{n-1}\cos n\theta
\]

\[
u_{\theta} = -nUR^{n-1}\sin n\theta
\]

Then, for \(0 < \theta < (\pi/2n)\), \(u_{r}\) is positive while \(u_{\theta}\) is negative and for \((\pi/2n) < \theta < (\pi/n)\), \(u_{r}\) is negative and \(u_{\theta}\) remains negative. This establishes the flow directions as indicated in Fig. 4.5.

From the foregoing, the complex potential for the flow in a corner or sector of angle \(\pi/n\) is

\[
F(z) = Uz^n
\]

(4.10)

For \(n = 1\), Eq. (4.10) gives the complex potential for a uniform rectilinear flow, and for \(n = 2\), it gives the complex potential for the flow in a right-angled corner.

### 4.6 FLOW AROUND A SHARP EDGE

The complex potential for the flow around a sharp edge, such as the edge of a flat plate, is obtained from the function \(z^{1/2}\). Then consider the complex potential

\[
F(z) = cz^{1/2}
\]

where \(c\) is real and \(0 < \theta < 2\pi\). Then, in cylindrical coordinates,

\[
F(z) = cR^{1/2}e^{i\theta/2}
\]

so that the velocity potential and stream function are

\[
\phi = cR^{1/2}\cos \frac{\theta}{2}
\]

\[
\psi = cR^{1/2}\sin \frac{\theta}{2}
\]

Thus the lines \(\theta = 0\) and \(\theta = 2\pi\) correspond to the streamline \(\psi = 0\). The other streamlines are defined by the equation \(R^{1/2}\sin \theta/2 = \) constant, which yields the flow pattern shown in Fig. 4.6. The direction of the flow is obtained from the complex velocity as follows:

\[
W(z) = \frac{c}{2z^{1/2}} = \frac{c}{2R^{1/2}e^{-i\theta/2}} = \frac{c}{2R^{1/2}} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)e^{-i\theta}
\]
Hence the velocity components are

\[ u_\theta = \frac{c}{2R^{1/2}} \cos \frac{\theta}{2} \]

\[ u_\phi = -\frac{c}{2R^{1/2}} \sin \frac{\theta}{2} \]

Then, for \( 0 < \theta < \pi \), \( u_\theta > 0 \) and \( u_\phi < 0 \). Also, for \( \pi < \theta < 2\pi \), \( u_\theta < 0 \) and \( u_\phi < 0 \). This gives the direction of flow as indicated in Fig. 4.6.

The flow field shown in Fig. 4.6 corresponds to the flow around a sharp edge, and so the complex potential for such a flow is

\[ F(z) = cz^{1/2} \]  \hspace{1cm} (4.11)

An important feature of this result is that the corner itself is a singular point at which the velocity components become infinite. Since both \( u_\theta \) and \( u_\phi \) vary as the inverse of \( R^{1/2} \), it follows that the velocity is singular as the square root of the distance from the edge. This result will be discussed in Sec. 4.15.

4.7 FLOW DUE TO A DOUBLET

The function \( 1/z \) has a singularity at \( z = 0 \), and in the context of complex potentials, this singularity is called a doublet. The quickest way of establishing the flow field which corresponds to the complex potentials which are proportional to \( 1/z \) would be to follow the methods which were used in the previous sections. However, it turns out that the doublet may be considered to be the coalescing of a source and a sink, and the required complex potential may be obtained through a limiting procedure which uses this fact. This interpretation leads to a better physical understanding of the doublet, and for this reason it will be followed here before studying the flow field.

Referring to the geometry indicated in Fig. 4.7a, consider a source of strength \( m \) and a sink of strength \( m \), each of which is located on the real axis a small distance \( e \) from the origin. The complex potential for such a configuration is, from Eq. (4.8),

\[ F(z) = \frac{m}{2\pi} \log \left( \frac{z + e}{z - e} \right) \]

\[ = \frac{m}{2\pi} \log \left( \frac{z + e}{z - e} \right) \]

\[ = \frac{m}{2\pi} \log \left( \frac{1 + e/z}{1 - e/z} \right) \]

If the nondimensional distance \( e/|z| \) is considered to be small, the argument of

It is now proposed to let \( e \to 0 \) and \( m \to \infty \) in such a way that \( \lim_{e \to 0}(me) = \pi \mu \), where \( \mu \) is a constant. Then the complex potential becomes

\[ F(z) = \frac{\mu}{z} \]

Thus the complex potential \( \mu/z \) may be thought of as being the equivalent of the superposition of a very strong source and a very strong sink which are very close together.
In order to establish the flow field which the above complex potential represents, the stream function will be established as follows:

\[ F(z) = \frac{\mu}{z + iy} \]
\[ = \frac{x - iy}{x^2 + y^2} \]
\[ \therefore \psi = -\frac{\mu y}{x^2 + y^2} \]

Thus the equation of the streamlines \( \psi = \text{constant} \) is

\[ x^2 + y^2 + \frac{\mu}{\psi} y = 0 \]

or

\[ x^2 + \left( y + \frac{\mu}{2\psi} \right)^2 = \left( \frac{\mu}{2\psi} \right)^2 \]

But this is the equation of a circle of radius \( \mu/(2\psi) \) whose center is located at \( y = -\mu/(2\psi) \). This gives the streamline pattern shown in Fig. 4.7b. Although the direction of the flow along the streamlines may be deduced from the source and sink interpretation, it will be checked by evaluating the velocity components. The complex velocity for this complex potential is

\[ W(z) = -\frac{\mu}{z^2} = -\frac{\mu}{R^2} e^{-i\theta} \]
\[ = -\frac{\mu}{R^2} \cos\theta - i \sin\theta e^{-i\theta} \]

Hence the velocity components are

\[ u_r = -\frac{\mu}{R^2} \cos\theta \]
\[ u_\theta = -\frac{\mu}{R^2} \sin\theta \]

These expressions for \( u_r \) and \( u_\theta \) confirm the flow directions indicated in Fig. 4.7b.

The flow field illustrated in Fig. 4.7b is called a doublet flow, and the singularity which is at the heart of the flow field is called a doublet. Then the complex potential for a doublet of strength \( \mu \) which is located at \( z = z_0 \) is

\[ F(z) = \frac{\mu}{z - z_0} \quad (4.12) \]

The principal use of the doublet is in the superposition of fundamental flow fields to generate more complex and more practical flow fields. An application of this will be illustrated in the next section.

4.8 CIRCULAR CYLINDER WITHOUT CIRCULATION

The fundamental solutions to the foregoing flow situations provide the basis for more general solutions through the principle of superposition. Superposition is valid here, since the governing equation, for either the velocity potential or the stream function, is linear. The first example of superposition of fundamental solutions will be the flow around a circular cylinder.

Consider the superposition of a uniform rectilinear flow and a doublet at the origin. Then, from Eqs. (4.7a) and (4.12), the complex potential for the resulting flow field will be

\[ F(z) = Uz + \frac{\mu}{z} \]

It will now be shown that for a certain choice of the doublet strength the circle \( R = a \) becomes a streamline. On the circle \( R = a \), the value of \( z \) is \( ae^{i\theta} \), so that the complex potential on this circle is

\[ F(z) = Uae^{i\theta} + \frac{\mu}{a} e^{-i\theta} \]
\[ = \left( Ua + \frac{\mu}{a} \right) \cos\theta + i \left( Ua - \frac{\mu}{a} \right) \sin\theta \]

Thus the value of the stream function on the circle \( R = a \) is

\[ \psi = \left( Ua - \frac{\mu}{a} \right) \sin\theta \]

For general values of \( \mu \), \( \psi \) is clearly variable, but if we choose the strength of the doublet to be \( \mu = Ua^2 \), then \( \psi = 0 \) on \( R = a \). The flow pattern for this doublet strength is shown in Fig. 4.8a. The flow field due to the doublet encounters that due to the uniform flow and is bent downstream. For clarity, the flow due to the doublet is shown dotted in Fig. 4.8a. It may be seen that the doublet flow is entirely contained within the circle \( R = a \), while the uniform flow is deflected by the doublet in such a way that it is entirely outside the circle \( R = a \). The circle \( R = a \) itself is common to the two flow fields.

Under these conditions, a thin metal cylinder of radius \( a \) could be slid into the flow field perpendicular to the uniform flow so that it coincides with the streamline on \( R = a \). Clearly the flow due to the doublet and that due to the free stream would be undisturbed by such a cylindrical shell. Having done this, the flow due to the doublet could be removed and the outer flow would remain unchanged. Finally, the inside of the shell could be filled to yield a solid cylinder. That is, for \( R \geq a \), the flow field due to the doublet of strength \( Ua^2 \) and the uniform rectilinear flow of magnitude \( U \) give the same flow as that for a uniform flow of magnitude \( U \) past a circular cylinder of radius \( a \). The latter flow is shown in Fig. 4.8b. Then the complex potential for a uniform flow of
magnitude $U$ past a circular cylinder of radius $a$ is

$$F(z) = U \left( z + \frac{a^2}{z} \right)$$

(4.13)

This result is useful in its own right, but it will also be found useful in later sections, through the technique of conformal transformations, to obtain additional solutions.

The solution given by Eq. (4.13) for the flow around a circular cylinder predicts no hydrodynamic force acting on the cylinder. This statement will be proved quantitatively in a later section, and in the meantime it will be proved qualitatively. Referring to Fig. 4.8b, it can be seen that the flow is symmetric about the $x$ axis. That is, for each point on the upper surface there is a corresponding point on the lower surface, vertically below it, for which the magnitude of the velocity is the same. Then, from the Bernoulli equation, the magnitude of the pressure is the same at these two points. Hence, by integrating $pdv$ around the surface of the cylinder, the lift force acting on the cylinder must be zero. Similarly, owing to the symmetry of the flow about the $y$ axis, the drag force acting on the cylinder is zero.

Although the foregoing result does not agree with our physical intuition, the potential-flow solution for the circular cylinder, and indeed for other bodies, is valuable. The absence of any hydrodynamic force on the cylinder is due to the neglect of viscosity. It will be seen in Part III that viscous effects create a thin boundary layer around the cylinder, and this boundary layer separates from the surface at some point, creating a low-pressure wake. The resulting pressure distribution creates a drag force. However, it will be pointed out that the viscous boundary-layer solution is valid only in the thin boundary layer around the cylinder, and the solution obtained from the boundary-layer equations must be matched to that given by Eq. (4.13) at the edge of the boundary layer. That is, Eq. (4.13) gives a valid solution outside the thin boundary layer and upstream of the vicinity of the separation point. It also indicates the idealized flow situation which would be approached if viscous effects are minimized. For more streamlined bodies, such as airfoils, the potential-flow solution is approached over the entire length of the body.

4.9 CIRCULAR CYLINDER WITH CIRCULATION

The flow field studied in the previous section not only was irrotational but it produced no circulation around the cylinder itself. It was found that there was no hydrodynamic force acting on the cylinder under these conditions. It will be shown in a later section that it is the circulation around a body which produces any lift force which acts on it. It is therefore of interest to study the flow around a circular cylinder which has a circulation around it.

It was established in a previous section that the streamlines for a vortex flow form concentric circles. Therefore, if a vortex was added at the origin to the flow around a circular cylinder, as described in the previous section, the fact that the circle $R = a$ was a streamline would be unchanged. Thus, from Eqs. (4.13) and (4.9), $z_0$ being zero in the latter, the complex potential for the flow around a circular cylinder with a negative bound vortex around it will be

$$F(z) = U \left( z + \frac{a^2}{z} \right) + \frac{i \Gamma}{2\pi} \log z + c$$

The negative vortex has been used, since it will turn out that this leads to a positive lift. A constant $c$ has been added to the complex potential for the following reason. For no circulation, it was found that not only was $\psi$ constant on $R = a$ but the value of the constant was zero. By adding the vortex, $\psi$ will no longer be zero on $R = a$, although it will have some other constant value. Since it is frequently useful to have the streamline on $R = a$ be $\psi = 0$, it is desirable to adjust things so that this condition is achieved. By adding a constant $c$ to the complex potential, we have the flexibility to choose $c$ in such a way that $\psi = \text{constant}$ becomes $\psi = 0$. It should be noted that this adjustment has no effect on the velocity and pressure distributions, since the velocity components are defined by derivatives of $\psi$, so that the absolute value of $\psi$ at any point is of no significance.

In order to evaluate the constant $c$, the value of the stream function on the circle $R = a$ will be computed. Then, putting $z = ae^{i\theta}$, the complex potential becomes

$$F(z) = U(\text{ae}^{i\theta} + \text{ae}^{-i\theta}) + \frac{i \Gamma}{2\pi} \log \text{ae}^{i\theta} + c$$

$$= 2Ua \cos \theta - \frac{\Gamma}{2\pi} \theta + \frac{i \Gamma}{2\pi} \log a + c$$
Hence on the circle \( R = a \) the value of \( \psi \) is indeed constant, and by choosing 
\[ c = -(i\Gamma/2\pi) \log a, \]
the value of this constant will be zero. With this value of \( c \), the complex potential becomes
\[
F(z) = U \left( z + \frac{a^2}{z} \right) + \frac{i\Gamma}{2\pi} \log \frac{z}{a}
\]
(4.14)

which describes a uniform rectilinear flow of magnitude \( U \) approaching a circular cylinder of radius \( a \) which has a negative vortex of strength \( \Gamma \) around it. As required, this result agrees with Eq. (4.13) when \( \Gamma = 0 \).

In order to visualize the flow field described by Eq. (4.14), the corresponding velocity components will be evaluated from the complex velocity.

\[
W(z) = U \left( 1 - \frac{a^2}{z^2} \right) + \frac{i\Gamma}{2\pi} \frac{1}{z}
\]
\[
= U \left( 1 - \frac{a^2}{R^2} e^{-i2\theta} \right) + \frac{i\Gamma}{2\pi R} e^{-i\theta}
\]
\[
= \left[ U \left( e^{i\theta} - \frac{a^2}{R^2} e^{-i\theta} \right) + \frac{i\Gamma}{2\pi R} \right] e^{-i\theta}
\]
\[
= \left\{ U \left( 1 - \frac{a^2}{R^2} \right) \cos \theta + i \left( U \left( 1 + \frac{a^2}{R^2} \right) \sin \theta + \frac{\Gamma}{2\pi R} \right) \right\} e^{-i\theta}
\]

Hence, by comparison with Eq. (4.6), the velocity components are
\[
u_R = U \left( 1 - \frac{a^2}{R^2} \right) \cos \theta \quad (4.15a)
\]
\[
u_\theta = -U \left( 1 + \frac{a^2}{R^2} \right) \sin \theta - \frac{\Gamma}{2\pi R} \quad (4.15b)
\]

On the surface of the cylinder, where \( R = a \), Eqs. (4.15) become
\[
u_R = 0
\]
\[
u_\theta = -2U \sin \theta - \frac{\Gamma}{2\pi a}
\]

The fact that \( \nu_R = 0 \) on \( R = a \) is to be expected, since this is the boundary condition (II.3). A significant point in the flow field is a point where the velocity components all vanish—that is, a stagnation point. For this flow field the stagnation points are defined by
\[
\sin \theta_s = -\frac{\Gamma}{4\pi U a}
\]
(4.16)

where \( \theta_s \) is the value of \( \theta \) corresponding to the stagnation point. For \( \Gamma = 0 \),

\[
\sin \theta_s = 0,
\]
so that \( \theta_s = 0 \) or \( \pi \), which agrees with Fig. 4.8b for the circular cylinder without circulation. For nonzero circulation, the value of \( \theta_s \) clearly depends upon the magnitude of the parameter \( \Gamma/(4\pi U a) \), and it is convenient to discuss Eq. (4.16) for different ranges of this parameter.

First, consider the range \( 0 < \Gamma/(4\pi U a) < 1 \). Here \( \sin \theta_s < 0 \), so that \( \theta_s \) must lie in the third and fourth quadrants. There are two stagnation points, and clearly the one which was at \( \theta = \pi \) is now located in the third quadrant while the one which was located at \( \theta = 0 \) is now located in the fourth quadrant. The two stagnation points will be symmetrically located about the \( y \) axis in order that \( \sin \theta_s = -\text{constant} \) may be satisfied. The resulting flow situation is shown in Fig. 4.9a.

Physically, the location of the stagnation points may be explained as follows: The flow due to the vortex and that due to the flow around the cylinder without circulation reinforce each other in the first and second quadrants. On the other hand, these two flow fields oppose each other in the third and fourth quadrants, so that at some point in each of these regions the net velocity is zero. Thus the effect of circulation around the cylinder is to make the front and rear stagnation points approach each other, and for a negative vortex they do so along the lower surface of the cylinder.

Consider next the case when the nondimensional circulation is unity, that is, when \( \Gamma/(4\pi U a) = 1 \). Here \( \sin \theta_s = -1 \), so that \( \theta_s = 3\pi/2 \). The corresponding flow configuration is shown in Fig. 4.9b. The two stagnation points have been brought together by the action of the bound vortex such that they coincide to form a single stagnation point at the bottom of the cylindrical surface. It is evident that if the circulation is increased above this value, the single stagnation point cannot remain on the surface of the cylinder. It will move off into the fluid as either a single stagnation point or two stagnation points.

\[\text{FIGURE 4.9} \]
Flow of approach velocity \( U \) around a circular cylinder of radius \( a \) with negative bound circulation of magnitude \( \Gamma \) for (a) \( 0 < \Gamma/(4\pi U a) < 1 \), (b) \( \Gamma/(4\pi U a) = 1 \), and (c) \( \Gamma/(4\pi U a) > 1 \).
Finally, consider the case where $\Gamma / (4\pi U a) > 1$. Since it seems likely that any stagnation points there may be will not lie on the surface of the cylinder, the velocity components must be evaluated from Eqs. (4.15). Then if $R_*$ and $\theta_*$ are the cylindrical coordinates of the stagnation points, it follows from Eqs. (4.15) that $R_*$ and $\theta_*$ must satisfy the equations

$$U \left(1 - \frac{a^2}{R_*^2}\right) \cos \theta_* = 0$$

$$U \left(1 + \frac{a^2}{R_*^2}\right) \sin \theta_* = -\frac{\Gamma}{2\pi R_*}$$

Since it is assumed that the stagnation points are not on the surface of the cylinder, it follows that $R_* \neq a$, so that the first of these equations requires that $\theta_* = \pi/2$ or $3\pi/2$. For these values of $\theta_*$, the second of the above equations becomes

$$U \left(1 + \frac{a^2}{R_*^2}\right) = \pm \frac{\Gamma}{2\pi R_*}$$

where the minus sign corresponds to $\theta_* = \pi/2$ and the plus sign to $\theta_* = 3\pi/2$. Since $U > 0$, the left-hand side of the above equation is positive, and since $\Gamma > 0$, the right-hand side must be rejected on the right-hand side. This might have been expected, since for $\Gamma / (4\pi U a) = 1$ the value of $\theta_*$ was $3\pi/2$, whereas the minus sign corresponds to $\theta_* = \pi/2$, which would require a large jump in $\theta_*$ for a small change in $\Gamma$. The equation for $R_*$ now becomes

$$U \left(1 + \frac{a^2}{R_*^2}\right) = \frac{\Gamma}{2\pi R_*}$$

or

$$R_*^2 - \frac{\Gamma}{2\pi U} R_* + a^2 = 0$$

hence

$$R_* = \frac{\Gamma}{4\pi U} \pm \sqrt{\left(\frac{\Gamma}{4\pi U}\right)^2 - a^2}$$

or

$$\frac{R_*}{a} = \frac{\Gamma}{4\pi U a} \left[1 \pm \sqrt{1 - \left(\frac{4\pi U a}{\Gamma}\right)^2}\right]$$

This result shows that as $4\pi U a / \Gamma \to 0$, $R_* \to \infty$ for the plus sign, but the corresponding limit is indeterminate for the minus sign. This difficulty may be overcome by expanding the square root for $4\pi U a / \Gamma \ll 1$ as follows:

$$\frac{R_*}{a} = \frac{\Gamma}{4\pi U a} \left[1 \pm \left(1 - \frac{1}{2} \left(\frac{4\pi U a}{\Gamma}\right)^2 + \cdots\right)\right]$$

where the dots indicate terms of order $(4\pi U a / \Gamma)^4$ or smaller. In this form it is evident that as $4\pi U a / \Gamma \to 0$, $R_* \to 0$ for the minus sign. Since this stagnation point would be inside the surface of the cylinder, the minus sign may be rejected, so that the coordinates of the stagnation point in the fluid outside the cylinder are

$$\theta_* = \frac{3\pi}{2}$$

$$\frac{R_*}{a} = \frac{\Gamma}{4\pi U a} \left[1 \pm \sqrt{1 - \left(\frac{4\pi U a}{\Gamma}\right)^2}\right]$$

This gives a single stagnation point below the surface of the cylinder. The corresponding flow configuration is shown in Fig. 4.9c, from which it will be seen that there is a portion of the fluid which perpetually encircles the cylinder.

The flow fields for the circular cylinder with circulation, as shown in Fig. 4.9, exhibit symmetry about the $y$ axis. Then, following the arguments used in the previous section, it may be concluded that there will be no drag force acting on the cylinder. However, the existence of the circulation around the cylinder has destroyed the symmetry about the $x$ axis; so there will be some force acting on the cylinder in the vertical direction. For the negative circulation shown the velocity on the top surface of the cylinder will be higher than that for no circulation, while the velocity on the bottom surface will be lower. Then, from Bernoulli’s equation, the pressure on the top surface will be lower than that on the bottom surface, so that the vertical force acting on the cylinder will be upward. That is, a positive lift will exist. In order to determine the magnitude of this lift, a quantitative analysis must be performed, and this will be done in the next section.

The principal interest in the flow around a circular cylinder with circulation is in the study of airfoil theory. By use of conformal transformations the flow around certain airfoil shapes may be transformed into that of the flow around a circular cylinder with circulation.

### 4.10 BLASIUS’ INTEGRAL LAWS

In the previous section it was argued that a lift force exists on a circular cylinder which has a circulation around it. However, the magnitude of the force can be established only by quantitative methods. The obvious way to evaluate the magnitude of this force is to establish the velocity components from the complex potential. Knowing the velocity components, the pressure distribution around the surface of the cylinder may be established by use of the Bernoulli equation. Integration of this pressure distribution will then yield the required force acting on the cylinder.
for the $x$ direction may be expressed by the following equation:

$$-X - \int_{C_0} p \, dy = \int_{C_0} \rho (u \, dy - v \, dx) \, u$$

In writing this equation, it has been noted that there is no transfer of momentum across the surface $C_0$, since it is a streamline, and that the integral of the pressure around $C_0$ yields the force $X$, which acts in the positive direction on the body and hence in the negative direction on the fluid. Also, the mass efflux across the element of the surface $C_0$ is $\rho (u \, dy - v \, dx)$, so that the product of this quantity and the $x$ component of velocity, when integrated around the surfaces $C_0$, gives the net increase in the $x$ component of momentum.

A similar equation may be obtained by applying the same newtonian law to the $y$ direction. Thus, the statement that the net external force acting in the positive $y$ direction must equal the net rate of increase of the $y$ component of the momentum yields the equation

$$-Y + \frac{1}{2} \int_{C_0} p \, dx = \int_{C_0} \rho (u \, dy - v \, dx) \, v$$

Solving these two equations for the unknown forces $X$ and $Y$ yields the following pair of integrals:

$$X = \int_{C_0} (-p \, dy - \rho u^2 \, dy + \rho v \, dx)$$

$$Y = \int_{C_0} (p \, dx - \rho u \, dy + \rho v^2 \, dx)$$

The pressure may be eliminated from these equations by use of the Bernoulli equation, which for the case under consideration, may be written in the form

$$p + \frac{1}{2} \rho (u^2 + v^2) = B$$

where $B$ is the Bernoulli constant. Then, by eliminating the pressure $p$, the expressions for $X$ and $Y$ become

$$X = \rho \int_{C_0} \left[ u \, v \, dx - u^2 \, dy \right]$$

$$Y = -\rho \int_{C_0} \left[ u \, v \, dy + \frac{1}{2} (u^2 - v^2) \, dx \right]$$

where the fact that $\int_{C_0} B \, dx = \int_{C_0} B \, dy = 0$ for any constant $B$ around any closed contour $C_0$ has been used.
It will now be shown that the quantity \( X - iY \) may be related to a complex integral. Consider the following complex integral involving the complex velocity \( W \):

\[
i \frac{\rho}{2} \int_{C_0} W^2 \, dz = i \frac{\rho}{2} \int_{C_0} (u - \omega^2)(dx + i\, dy)
\]

\[
= i \frac{\rho}{2} \int_{C_0} \left[ \left( (u^2 - v^2) \, dx + 2uv \, dy \right) + i \left( (u^2 - v^2) \, dy - 2uv \, dx \right) \right]
\]

\[
= \rho \int_{C_0} \left[ (ux \, dx - \frac{1}{2}(u^2 - v^2) \, dy) + i (uy \, dy + \frac{1}{2}(u^2 - v^2) \, dx) \right]
\]

\[= X - iY\]

The last equality follows by comparison of the expanded form of the complex integral with the expressions derived above for the body forces \( X \) and \( Y \). That is, the complex force \( X - iY \) may be evaluated from

\[X - iY = i \frac{\rho}{2} \int_{C_0} W^2 \, dz \quad (4.17a)\]

where \( W(z) \) is the complex velocity for the flow field and \( C_0 \) is any closed contour which encloses the body under consideration. It should be noted that \( X \) and \( Y \) were defined as the forces acting on the body through its center of gravity.

Equation (4.17a) constitutes one of the two Blasius laws. Normally, in applying Eq. (4.17a), the contour integral is evaluated with the aid of the residue theorem. An application of this procedure will be covered in the next section.

In order to establish the hydrodynamic moment acting on the body, consider again Fig. 4.10. The quantity \( M \) is the moment acting on the body about its center of gravity. Then, taking clockwise moments as positive, moment equilibrium of the fluid enclosed between \( C_0 \) and \( C_1 \) requires that

\[-M + \oint_{C_0} \left[ px \, dx + py \, dy + \rho (u \, dy - v \, dx) \, uy - \rho (u \, dy - v \, dx) \, ux \right] = 0\]

The first two terms under the integral are the components of the pressure force multiplied by their respective perpendicular distances from the center of gravity of the body, which is at the origin of the coordinate system. The remaining two terms under the integral represent the inertia forces, which were evaluated in the discussion of the force equations, multiplied by their respective perpendicular distances from the origin. These inertia forces are equal in magnitude and opposite in direction to the rate of increase of the horizontal and vertical momentum components.

Solving the foregoing equation for the hydrodynamic moment \( M \) gives

\[M = -\frac{\rho}{2} \int_{C_0} \left[ px \, dx + py \, dy + \rho (u^2 \, dy + v^2 \, dx - u\, wy \, dx - u\, wx \, dy) \right]\]

Substituting \( p = B - \rho(u^2 + v^2)/2 \) from the Bernoulli equation gives

\[M = \rho \int_{C_0} \left[ \frac{1}{2} (u^2 + v^2) (x \, dx + y \, dy) + (u^2 y \, dy + v^2 x \, dx) - (u\, wx \, dx + u\, wy \, dy) \right]\]

where the fact has been used that \( \oint_{C_0} B \, dx = \oint_{C_0} B \, dy = 0 \) for any constant \( B \) and any closed contour \( C_0 \). Rearranging the above equation shows that the integral for the moment \( M \) may be put in the following form:

\[M = -\frac{\rho}{2} \oint_{C_0} \left[ (u^2 - v^2) (x \, dx - y \, dy) + 2uv (x \, dy + y \, dx) \right]\]

It will now be shown that the quantity \( M \) may be related to the real part of a complex integral. Consider the real part, designated by \( \text{Re} \{ \} \), of the following complex integral:

\[\text{Re} \left( \frac{\rho}{2} \oint_{C_0} zW^2 \, dz \right) = \text{Re} \left( \frac{\rho}{2} \oint_{C_0} (x + iy)(u - \omega^2)(dx + i\, dy) \right)\]

\[= \text{Re} \left( \frac{\rho}{2} \oint_{C_0} \left[ (u^2 - v^2)(x \, dx - y \, dy) + 2uv (x \, dy + y \, dx) \right] + i \frac{\rho}{2} \oint_{C_0} \left[ (u^2 - v^2)(x \, dx + y \, dy) - 2uv (x \, dx - y \, dy) \right] \right)\]

\[= \frac{\rho}{2} \oint_{C_0} \left[ (u^2 - v^2)(x \, dx - y \, dy) + 2uv (x \, dy + y \, dx) \right] \]

\[= -M \]

The last equality follows from a comparison of the real part of the complex integral with the expression derived for \( M \). That is, the hydrodynamic moment acting on a body is given by

\[M = -\frac{\rho}{2} \text{Re} \left( \oint_{C_0} zW^2 \, dz \right) \quad (4.17b)\]

where \( W(z) \) is the complex velocity for the flow field and \( C_0 \) is any closed contour which encloses the body. It should be noted that \( M \) is defined as the hydrodynamic moment acting on the body, and it is positive when it acts in the clockwise direction. Equation (4.17b) is the second of the Blasius laws, and the contour integral in this equation is usually evaluated by use of the residue theorem.
4.11 FORCE AND MOMENT ON A CIRCULAR CYLINDER

It was observed in an earlier section that a force exists on a circular cylinder which is immersed in a uniform flow and which has a circulation around it. The magnitude of this force may now be evaluated using the results of the previous section.

From Eq. (4.14), the complex potential for a circular cylinder of radius $a$ in a uniform rectilinear flow of magnitude $U$ and having a bound vortex of magnitude $\Gamma$ in the negative direction is

$$ F(z) = U\left(1 + \frac{z^2}{a^2}\right) + \frac{i\Gamma}{2\pi} \log \frac{z}{a} $$

Then the complex velocity for this flow field is

$$ W(z) = U\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi z} $$

$$ \therefore \ W^2(z) = U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} + \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2} $$

But from the Blasius integral law [Eq. (4.17a)]

$$ X - iY = i\frac{\rho}{2} \int_C W^2 \, dz $$

$$ = i\frac{\rho}{2} \left[2\pi i \sum \text{(residues of } W^2 \text{ inside } C_0)\right] $$

where the last equality follows from the residue theorem. It is therefore required to evaluate the residue of $W^2(z)$ at each of the singular points which lie inside an arbitrary contour in the fluid which encloses the cylinder. But inspection of the expression derived for $W^2(z)$ above shows that the only singularity is at $z = 0$, corresponding to the doublet and the vortex which are located there. Furthermore, $W^2(z)$ is in the form of its Laurent series about $z = 0$, from which it is seen that the only term of the form $b/z$ is the fourth one. Hence, the residue of $W^2(z)$ at $z = 0$ is $iU\Gamma/\pi$. Then the value of the complex force is

$$ X - iY = i\frac{\rho}{2} \left[2\pi i \left(\frac{iU\Gamma}{\pi}\right)\right] $$

$$ = -i\rho U\Gamma $$

Equating the real and imaginary parts of this equation shows that the drag force $X$ is zero, as was expected, and that the value of the lift force is

$$ Y = \rho U\Gamma \quad (4.18) $$

Equation (4.18) is known as the Kutta-Joukowsky law, and it asserts that, for flow around a circular cylinder, there will be no lift force on the cylinder if there is no circulation around it, and if there is a circulation, the value of the lift force will be given by the product of the magnitude of this circulation with the free-stream velocity and the density of the fluid. It should be noted that the right-hand side of Eq. (4.18) is positive, so that the negative circulation which acted on the cylinder led to a positive, that is, upward, lift force.

In order to evaluate the hydrodynamic moment $M$ acting on the cylinder, the quantity $zW^3$ must be evaluated. From the expression for $W^2(z)$ which was established above,

$$ zW^2(z) = U^2z - \frac{2U^2a^2}{z} + \frac{U^2a^4}{z^3} + \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2} $$

But from the Blasius integral law [Eq. (4.17b)]

$$ M = -\frac{\rho}{2} \text{Re} \left( \int_C zW^2 \, dz \right) $$

$$ = -\frac{\rho}{2} \text{Re} \left[ 2\pi i \sum \text{(residues of } zW^2 \text{ inside } C_0)\right] $$

where again the residue theorem has been used. But the quantity $zW^2(z)$, as evaluated above, is already in the form of its Laurent series about $z = 0$. From this, it is evident that the only singularity is at $z = 0$, and the residue there comes from the second and last term in the expansion. Hence

$$ M = -\frac{\rho}{2} \text{Re} \left[ 2\pi i \left( -2U^2a^3 - \frac{\Gamma^2}{4\pi^2} \right) \right] $$

$$ = 0 $$

That is, as might be expected, there is no hydrodynamic moment acting on the cylinder.

4.12 CONFORMAL TRANSFORMATIONS

Many complicated flow boundaries may be transformed into regular flow boundaries, such as the ones already studied, by the technique of conformal transformations. Before using this fact, it is necessary to study the effect of conformal transformations on the complex potential, the complex velocity, sources, sinks, and vortices. In carrying out this study, it will be considered that some geometric shape in the $z$ plane whose coordinates are $x$ and $y$ is mapped into some other shape in the $\zeta$ plane whose coordinates are $\xi$ and $\eta$ by means of the transformation

$$ \zeta = f(z) $$

where $f$ is an analytic function. This situation is depicted in Fig. 4.11.

The basis of the complex potential was that both the velocity potential and the stream function had to satisfy Laplace's equation. Hence, in order to
plane, the sum of the above two quantities must be zero. Then
\[
\left( \frac{\partial \xi}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \xi}{\partial y} \right)^2 + 2 \left( \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} \right) + 2 \left( \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) = 0
\]

This is the equation which has to be satisfied by \( \phi(\xi, \eta) \) in the \( \xi \) plane due to any transformation \( \xi = f(x) \) corresponding to \( \frac{\partial \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \) in the \( x \) plane. So far, no restrictions have been imposed on the transformation. But if the transformation is conformal, the mapping function \( f \) will be analytic and the real and imaginary parts of the new variable \( \xi \) will be harmonic. That is, \( \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 \) and \( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0 \), so that the terms involving these quantities in the equation for \( \phi \) will be zero. Also, \( \xi(x, y) \) and \( \eta(x, y) \) must satisfy the Cauchy-Riemann equations if the mapping function is analytic. That is,
\[
\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}
\]
and
\[
\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x}
\]
then
\[
\left( \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) = 0
\]

Using this result, the equation to be satisfied by \( \phi \) becomes
\[
\left( \frac{\partial \xi}{\partial x} \right)^2 + \left( \frac{\partial \xi}{\partial y} \right)^2 + \left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 = 0
\]

Using the Cauchy-Riemann equations to eliminate first \( \xi \), then \( \eta \), then shows that the following pair of equations must be satisfied:
\[
\left( \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial \eta}{\partial y} \right)^2 = 0
\]
\[
\left( \frac{\partial \xi}{\partial x} \right)^2 + \left( \frac{\partial \xi}{\partial y} \right)^2 = 0
\]

But these equations must be satisfied for all analytic mapping functions; hence it follows that
\[
\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0
\]

That is, Laplace's equation in the \( x \) plane transforms into Laplace's equation in
the $\zeta$ plane, provided these two planes are related by a conformal transformation. Then, since both $\phi$ and $\psi$ must satisfy Laplace's equation, it follows that a complex potential in the $z$ plane is also a valid complex potential in the $\zeta$ plane, and vice versa. This means that if the solution for some simple body is known in one of these planes, say the $\zeta$ plane, then the solution for the more complex body may be obtained by substituting $\zeta = f(z)$ in the complex potential $F(\zeta)$.

Consider now what happens to the complex velocity under a conformal transformation. Starting in the $z$ plane with the definition of complex velocity,

$$W(z) = \frac{dF(z)}{dz} = \frac{d\zeta}{dz} \frac{dF(\zeta)}{d\zeta}$$  \hspace{1cm} (4.19)

That is, complex velocities are not, in general, mapped one to one, but they are proportional to each other, and the proportionality factor depends on the mapping function.

Finally, the effect of a conformal transformation on the strength of the basic singularities will be investigated. That is, the strength of transformed sources, sinks, and vortices will be established. This is most readily done by first proving the general relation that the integral of the complex velocity around any closed contour in the flow field equals $\Gamma + im$, where $\Gamma$ is the net strength of any vortices inside the contour and $m$ is the net strength of any sources and sinks inside the contour.

To prove this relation, consider any closed contour $C$ such as the one shown in Fig. 4.12. An element $dl$ of this contour is shown resolved into its coordinate components. Then the net strength of all the sources inside $C$ (sinks being considered negative sources) and the net strength of all the vortices inside $C$ will be given by

$$m = \int_C u \cdot n \, dl = \int_C (u \, dy - v \, dx)$$

$$\Gamma = \int_C u \cdot n \, dl = \int_C (u \, dx + v \, dy)$$

Now consider the integral around $C$ of the complex velocity $W(z)$.

$$\int_C W(z) \, dz = \int_C (u - iw)(dx + idy)$$

$$= \int_C (u \, dx + v \, dy) + i \int_C (u \, dy - v \, dx)$$

$$= \Gamma + im$$

where the last equality follows from a comparison with the expressions derived for $m$ and $\Gamma$. This general result will now be applied to a single vortex $\Gamma_z$ and a single source $m_z$ located in the $z$ plane. Then

$$\Gamma_z + im_z = \int_{C_z} W(z) \, dz$$

$$= \int_{C_z} W(\zeta) \frac{d\zeta}{dz} \, dz$$

$$= \int_{C_z} W(\zeta) \, d\zeta$$

$$= \Gamma_z + im_z$$

where $C_z$ is some closed contour in the $z$ plane and $C_z$ is its counterpart in the mapped plane. $\Gamma_z$ and $m_z$ are the corresponding vortex and source strengths in the $\zeta$ plane, and the above result shows that the vortex and source strengths are the same in the $\zeta$ plane as in the $z$ plane. That is, sources, sinks, and vortices map into sources, sinks, and vortices of the same strength under a conformal transformation.

In summary, if the complex potential for the flow around some body is known in the $\zeta$ plane, then the complex potential for the body corresponding to the conformal mapping $\zeta = f(z)$ may be obtained by substituting this transformation into the complex potential $F(\zeta)$. Complex velocities, on the other hand, do not transform one to one but are related by Eq. (4.19). Sources, sinks, and vortices maintain the same strength under conformal transformations.
4.13 JOUKOWSKI TRANSFORMATION

One of the most important transformations in the study of fluid mechanics is the Joukowski transformation. By means of this transformation and the basic flow solutions already studied, it is possible to obtain solutions for the flow around ellipses and a family of airfoils. The Joukowski transformation is of the form

\[ z = \zeta + \frac{c^2}{\zeta} \quad (4.20) \]

where the constant \( c^2 \) is usually taken to be real. A general property of the Joukowski transformation is that for large values of \( |\zeta| \), \( z \to \zeta \). That is, far from the origin the transformation becomes the identity mapping, so that the complex velocity in the two planes is the same far from the origin. This means that if a uniform flow of a certain magnitude is approaching a body in the \( z \) plane at some angle of attack, a uniform flow of the same magnitude and angle of attack will approach the corresponding body in the \( \zeta \) plane.

From Eq. (4.20),

\[ \frac{dz}{d\zeta} = 1 - \frac{c^2}{\zeta^2} \]

so that there is a singular point in the Joukowski transformation at \( \zeta = 0 \). Since we are normally dealing with the flow around some body, the point \( \zeta = 0 \) is normally not in the fluid, and so this singularity is of no consequence. There are also two critical points of the transformation, that is, points at which \( dz/d\zeta \) vanishes, at \( \zeta = \pm c \). Since smooth curves passing through critical points of a mapping may become corners in the transformed plane, it is of interest to investigate the consequence of a smooth curve passing through the critical points of the Joukowski transformation. To do this, consider an arbitrary point \( z \) and its counterpart \( \zeta \) as shown in Fig. 4.13a. Let the point \( \zeta \) be measured by the radii \( R_1 \) and \( R_2 \) and the angles \( \nu_1 \) and \( \nu_2 \) relative to the two critical points \( \zeta = c \) and \( \zeta = -c \), respectively. But according to the Joukowski transformation the points \( \zeta = \pm c \) map into the points \( z = \pm 2c \). Then let the mapping of the point \( \zeta \) be measured by the radii \( R_1 \) and \( R_2 \) and the angles \( \theta_1 \) and \( \theta_2 \) relative to the two points \( z = 2c \) and \( z = -2c \), respectively.

From Eq. (4.20),

\[ z + 2c = \frac{(\zeta + c)^2}{\zeta} \]

and

\[ z - 2c = \frac{(\zeta - c)^2}{\zeta} \]

\[ \therefore \quad \frac{z - 2c}{z + 2c} = \frac{\zeta - c}{\zeta + c} \]

Thus, with reference to Fig. 4.13a,

\[ \frac{R_1 e^{i\theta_1}}{R_2 e^{i\theta_2}} = \left( \frac{\rho_1 e^{i\nu_1}}{\rho_2 e^{i\nu_2}} \right)^2 \]

or

\[ \frac{R_1}{R_2} e^{i(\theta_1 - \theta_2)} = \left( \frac{\rho_1}{\rho_2} \right)^2 e^{2(\nu_1 - \nu_2)} \]

Equating the modulus and the argument of each side of this equation shows that

\[ \frac{R_1}{R_2} = \left( \frac{\rho_1}{\rho_2} \right)^2 \]

and

\[ \theta_1 - \theta_2 = 2(\nu_1 - \nu_2) \]

This last result shows that if a smooth curve passes through the point \( \zeta = c \), the corresponding curve in the \( z \) plane will form a knife-edge or cusp. This may be verified by considering a smooth curve to pass through the point \( \zeta = c \). Two points on this curve are shown in Fig. 4.13b, from which it is seen that \( \nu_1 \) changes from \( 3\pi/2 \) to \( \pi/2 \) and \( \nu_2 \) changes from \( 2\pi \) to \( 0 \) as the critical point is passed. That is, the value of \( \nu_1 - \nu_2 \) changes from \( -\pi/2 \) to \( \pi/2 \), giving a difference of \( \pi \). From the result \( \theta_1 - \theta_2 = 2(\nu_1 - \nu_2) \), it follows that the corresponding difference in the value of \( \theta_1 - \theta_2 \) will be \( 2\pi \). This yields a knife-edge or cusp in the \( z \) plane as shown in Fig. 4.13b. That is, if a smooth
curve passes through either of the critical points \( \zeta = \pm c \), the corresponding curve in the \( z \) plane will contain a knife-edge at the corresponding critical point \( z = \pm 2c \).

An example of a smooth curve which passes through both critical points is a circle centered at the origin of the \( \zeta \) plane and whose radius is \( c \), the constant which appears in the Joukowski transformation. Then, on this circle \( \zeta = ce^{i\nu} \), so that the value of \( z \) will be given by

\[
z = ce^{i\nu} + ce^{-i\nu} = 2c \cos \nu
\]

That is, the circle in the \( \zeta \) plane maps into the strip \( y = 0, x = 2c \cos \nu \) in the \( z \) plane. It is readily verified that all points which lie outside the circle \( |\zeta| = c \) cover the entire \( z \) plane. However, the points inside the circle \( |\zeta| = c \) also cover the entire \( z \) plane, so that the transformation is double-valued. This is readily verified by observing that for any value of \( \zeta \) Eq. (4.20) yields the same value of \( z \) for that value of \( \zeta \) and also for \( c^2/\zeta \). It will be noted that \( c^2/\zeta \) is simply the image of the point \( \zeta \) inside the circle of radius \( c \).

This double-valued property of the Joukowski transformation is treated by connecting the two points \( \zeta = \pm 2c \) by a branch cut along the \( x \) axis and creating two Riemann sheets. Then the mapping is single-valued if all the points outside the circle \( |\zeta| = c \) are taken to fall on one of these sheets and all the points inside the circle to fall on the other sheet. In fluid mechanics, difficulties due to the double-valued behavior do not usually arise because the points \( |\zeta| < c \) usually lie inside some body about which the flow is being studied, so that these points are not in the flow field in the \( z \) plane.

### 4.14 Flow Around Ellipses

Applications of the Joukowski transformation will be made in an inverse sense. That is, the simple geometry of the circle, the flow around which is known, will be placed in the \( \zeta \) plane, and the corresponding body which results in the \( z \) plane will be investigated by use of Eq. (4.20).

Consider, first, the constant \( c \) in Eq. (4.20) to be real and positive, and consider a circle of radius \( a > c \) to be centered at the origin of the \( \zeta \) plane. The contour in the \( z \) plane corresponding to this circle in the \( \zeta \) plane may be identified by substituting \( \zeta = ae^{i\nu} \) into Eq. (4.20).

\[
z = ae^{i\nu} + \frac{c^2}{a} e^{-i\nu} = \left( a + \frac{c^2}{a} \right) \cos \nu + i \left( a - \frac{c^2}{a} \right) \sin \nu
\]

Equating real and imaginary parts of this equation gives

\[
x = \left( a + \frac{c^2}{a} \right) \cos \nu
\]
\[
y = \left( a - \frac{c^2}{a} \right) \sin \nu
\]

These are the parametric equations of the required curve in the \( z \) plane. The equation of the curve may be obtained by eliminating \( \nu \) by use of the identity \( \cos^2 \nu + \sin^2 \nu = 1 \). This gives

\[
\left( \frac{x}{a + c^2/a} \right)^2 + \left( \frac{y}{a - c^2/a} \right)^2 = 1
\]

which is the equation of an ellipse whose major semiaxis is of length \( a + c^2/a \), aligned along the \( x \) axis, and whose minor semiaxis is of length \( a - c^2/a \). Then, in order to obtain the complex potential for a uniform flow of magnitude \( U \) approaching this ellipse at an angle of attack \( \alpha \), the same flow should be considered to approach the circular cylinder in the \( \zeta \) plane. But it is shown in Prob. 4.5, Eq. (4.29), that the complex potential for a uniform flow of magnitude \( U \) approaching a circular cylinder of radius \( a \) at an angle \( \alpha \) to the reference axis is

\[
F(\zeta) = U \left( \zeta e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{\zeta} \right)
\]

Then, by solving Eq. (4.20) for \( \zeta \) in terms of \( z \), the complex potential in the \( z \) plane may be obtained. From Eq. (4.20),

\[
\zeta^2 - z\zeta + c^2 = 0
\]

\[
\therefore \quad \zeta = \frac{z}{2} \pm \sqrt{\left( \frac{z}{2} \right)^2 - c^2}
\]

Since it is known that \( \zeta \to z \) for large values of \( z \), the positive root must be chosen. Then the complex potential in the \( z \) plane becomes

\[
F(z) = U \left[ \frac{z}{2} + \sqrt{\left( \frac{z}{2} \right)^2 - c^2} e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z/2 + \sqrt{(z/2)^2 - c^2}} \right]
\]

\[
= U \left[ z - \frac{z}{2} + \sqrt{(z/2)^2 - c^2} e^{-i\alpha} + \frac{a^2}{c^2} \frac{z}{2} - \sqrt{(z/2)^2 - c^2} e^{i\alpha} \right]
\]

where the last term has been rationalized. By writing \( z/2 \) as \( z - z/2 \) in the first
This gives the coordinates of the stagnation points as

\[
\begin{align*}
  x &= \pm \left( a + \frac{c^2}{a} \right) \cos \alpha \\
  y &= \pm \left( a - \frac{c^2}{a} \right) \sin \alpha
\end{align*}
\]

Equation (4.21a) includes two special cases within its range of validity. For \( \alpha = 0 \) it describes a uniform rectilinear flow approaching a horizontally oriented ellipse, and for \( \alpha = \pi/2 \) it describes a uniform vertical flow approaching the same horizontally oriented ellipse. However, it is of interest to note that the solution for a uniform rectilinear flow approaching a vertically oriented ellipse may be obtained directly from the Joukowski transformation with a slight modification. Substitute \( c = ib \), where \( b \) is real and positive, into Eq. (4.19).

\[
z = \frac{\frac{b^2}{c}}{\zeta}
\]

Then, as with the horizontal ellipse, examining the mapping of the circle \( \zeta = ae^{i\alpha} \) gives the parametric equations of the mapped boundary.

\[
\begin{align*}
  x &= \left( a - \frac{b^2}{a} \right) \cos \nu \\
  y &= \left( a + \frac{b^2}{a} \right) \sin \nu
\end{align*}
\]

Thus the equation of the contour in the \( z \) plane is

\[
\left( \frac{x}{a - \frac{b^2}{a}} \right)^2 + \left( \frac{y}{a + \frac{b^2}{a}} \right)^2 = 1
\]

which is the equation of an ellipse whose major semi-axis is \( a + \frac{b^2}{a} \) which is aligned along the \( y \) axis. Then to obtain a uniform rectilinear flow approaching such an ellipse the same flow should approach the circle in the \( \zeta \) plane. Thus the required complex potential, from Eq. (4.13), is

\[
F(\zeta) = \frac{U}{2} \left( \zeta + \frac{\zeta^2}{\zeta} \right)
\]

But the inverted equation of the mapping for which \( \zeta \rightarrow z \) as \( z \rightarrow \infty \) is

\[
\zeta = \frac{z}{2} + \sqrt{\left( \frac{z}{2} \right)^2 + b^2}
\]
Hence the complex potential in the $z$ plane is

$$F(z) = U \left[ \frac{z}{2} + \sqrt{ \left( \frac{z}{2} \right)^2 + b^2} \right] + \frac{a^2}{z / 2 + \sqrt{(z / 2)^2 + b^2}} \tag{4.21b}$$

$$F(z) = U \left[ z - \left( 1 + \frac{a^2}{b^2} \right) \left( \frac{z}{2} - \sqrt{\left( \frac{z}{2} \right)^2 + b^2} \right) \right]$$

in which the same rationalization and simplification has been carried out as before. Again the complex potential is in the form of that for a uniform flow plus a perturbation which is large near the body and which vanishes at large distances from the body. Equation (4.21b) describes a uniform rectilinear flow of magnitude $U$ approaching a vertically oriented ellipse. The flow field for this situation is shown in Fig. 4.14b.

### 4.15 KUTTA CONDITION AND THE FLAT-PLATE AIRFOIL

It was observed in Sec. 4.6 that the potential flow solution for flow around a sharp edge contained a singularity at the edge itself. This singularity required an infinite velocity at the point in question which, of course, is physically impossible. The question arises, then, as to what the real flow situation would be in a physical experiment. Depending upon the actual physical configuration, one of two remedial situations will prevail. One possibility is that the fluid will separate from the solid surface at the knife-edge. The resulting free streamline configuration would be such that the radius of curvature at the edge becomes finite rather than being zero. As a consequence, the velocities there will remain finite. Examples of this type of solution will be discussed later in this chapter.

A second possibility is that a stagnation point exists at the sharp edge. For the flow around finite bodies, stagnation points exist, and it seems possible that a stagnation point could be induced by the flow field to move to the location of the sharp edge. This possibility leads to the so-called “Kutta condition,” and it will be discussed below in the context of the flat-plate airfoil—that is, a flat plate which is at some angle of attack to the free stream.

In the previous section, the flow around an ellipse was obtained from the Joukowski transformation [Eq. (4.20)] by considering the flow around a circular cylinder of radius $a > c$ in the $\zeta$ plane. Now, if the constant $c$ is allowed to approach the magnitude of the radius $a$, the resulting ellipse in the $z$ plane degenerates to a flat plate defined by the strip $-2a \leq x \leq 2a$. The resulting flow field, as defined by Eq. (4.21a), is shown in Fig. 4.15a. Because of the angle of attack, the stagnation points do not coincide with the leading and trailing edges of the flat plate. Rather, the upstream stagnation point is located on the lower surface and the downstream stagnation point is located on the upper surface at the points $x = \pm 2a \cos \alpha$. Then, around both the leading and trailing edges, the flow will be that associated with a sharp edge which was discussed in Sec. 4.6. In that section, it was observed that infinite velocity components existed at the edge itself, a situation which is physically impossible to realize.

The difficulty encountered above with the flat-plate airfoil does not occur at the leading edge of real airfoils because real airfoils have a finite thickness and so have a finite radius of curvature at the leading edge. However, the trailing edge of airfoils is usually quite sharp, so that the difficulty of infinite velocity components still exists there. However, this remaining difficulty would also be overcome if the stagnation point which is near the trailing edge was actually at the trailing edge. This would be accomplished if a circulation existed around the flat plate and the magnitude of this circulation was just the amount required to rotate the rear stagnation point so that its location coincides with the trailing edge. This condition is called the Kutta condition, and it may be restated as follows: For bodies with sharp trailing edges which are at small angles of attack to the free stream, the flow will adjust itself in such a way that the rear stagnation point coincides with the trailing edge.

The amount of circulation required to comply with the Kutta condition may be determined as follows: In the $\zeta$ plane of Fig. 4.15a, the rear stagnation point is located at the point $\zeta = a e^{i\alpha}$. But, according to the Kutta condition, the rear stagnation point should be located at the point $\zeta = 2a$, which corresponds
to the point $\zeta = a$. That is, the stagnation point on the downstream face of the circular cylinder in the $\zeta$ plane should be rotated clockwise through an angle $\alpha$. But from Eq. (4.16), the magnitude of the circulation which will do this is

$$\Gamma = 4\pi Ua \sin \alpha$$

(4.22a)

in the clockwise direction (that is, negative circulation). Then the complex potential for the required flow in the $\zeta$ plane is, from Eqs. (4.14) and (4.29),

$$F(\zeta) = U \left( \zeta e^{-ia} + \frac{a^2}{\zeta} e^{ia} \right) + i 2Ua \sin \alpha \log \frac{\zeta}{a}$$

But the equation of the mapping is

$$z = \zeta + \frac{a^2}{\zeta}$$

and the inverse which gives $\zeta \to z$ as $z \to \infty$ is

$$\zeta = \frac{z}{2} + \sqrt{\left( \frac{z}{2} \right)^2 - a^2}$$

Then the complex potential in the $z$ plane is

$$F(z) = U \left\{ \frac{z}{2} + \sqrt{\left( \frac{z}{2} \right)^2 - a^2} e^{-ia} + \frac{a^2 e^{ia}}{z/2 + \sqrt{\left( z/2 \right)^2 - a^2}} \right\}$$

$$+ i2Ua \sin \alpha \log \left\{ \frac{1}{a} \left[ \frac{z}{2} + \sqrt{\left( \frac{z}{2} \right)^2 - a^2} \right] \right\}$$

(4.22b)

The flow field corresponding to this complex potential is shown in Fig. 4.15b. Although the flow at the trailing edge is now regular, the singularity at the leading edge still exists. In an actual flow configuration the fluid would separate at the leading edge and reattach again on the top side of the airfoil. The streamline $\psi = 0$ would then correspond to a finite curvature, and the velocity components would remain finite at the leading edge.

The lift force generated by the flat-plate airfoil may be calculated from the Kutta-Joukowski law. Then, denoting the lift force by $Y$ and using the value of the circulation given by Eq. (4.22a),

$$Y = 4\pi \rho U^2 a \sin \alpha$$

It is usual to express lift forces in terms of the dimensionless lift coefficient $C_L$, which is defined as follows:

$$C_L = \frac{Y}{\frac{1}{2} \rho U^2 l}$$

where $l$ is the length or chord of the airfoil which, for the flat plate under consideration, equals $4a$. Then the value of the lift coefficient for the flat-plate airfoil is

$$C_L = 2\pi \sin \alpha$$

This result shows that the lift coefficient for the flat-plate airfoil increases with angle of attack, and for small values of $\alpha$, for which $\sin \alpha = \alpha$, the lift coefficient is proportional to the angle of attack with a constant of proportionality of $2\pi$. This result is very close to experimental observations, and so the Kutta condition appears to be well justified. If the Kutta condition were not valid, there would be no circulation around the flat plate, and consequently, no lift would be generated. This would mean that kites would not be able to fly.

### 4.16 SYMMETRICAL JOUKOWSKI AIRFOIL

A family of airfoils may be obtained in the $z$ plane by considering the Joukowski transformation in conjunction with a series of circles in the $\zeta$ plane whose centers are slightly displaced from the origin. These airfoils are known as the Joukowski family of airfoils. Consider, first, the case where the center of the circle in the $\zeta$ plane is displaced from the origin along the real axis. It must then be decided in which direction the center should be moved and what radius should be employed, relative to the Joukowski constant $c$. From previous sections it is known that if the circumference of the circle passes through either of the two critical points of the Joukowski transformation, $\zeta = \pm c$, then a sharp edge or cusp is obtained in the $z$ plane. Then, if the leading edge of the airfoil is to have a finite radius of curvature and if there should be no singularities in the flow field itself, it follows that the point $\zeta = -c$ should be inside the circle in the $\zeta$ plane. Also, since the trailing edge of the airfoil should be sharp as opposed to being blunt, the circumference of the circle should pass through the point $\zeta = c$. These conditions will be satisfied by taking the center of the circle to be at $\zeta = -m$, where $m$ is real, and by choosing the radius of the circle to be $c + m$. Such a configuration is shown in Fig. 4.16a. The radius $a$ is given by

$$a = c + m = c(1 + e)$$

where the parameter $e = m/c$ will be assumed to be small compared with unity. When $e = 0$, the flat-plate airfoil is recovered, so that for $e \ll 1$ it may be anticipated that a thin airfoil will be obtained. The significance of the restriction $e \ll 1$ will be that all the equations may be linearized in $e$, which will permit a closed-form solution for the equation of the airfoil surface in the $z$ plane. Also shown in Fig. 4.16a is the airfoil which is obtained in the $z$ plane and its principal parameters, the chord $l$ and the maximum thickness $t$. It is now required to relate these parameters to the free parameters $a$ and $m$ and to establish the equation of the airfoil surface in the $\zeta$ plane.

To establish the chord of the airfoil in terms of the chosen radius $a$ and offset $m$, it is only necessary to find the mapping of the points $\zeta = c$ and $\zeta = -(c + 2m)$, since these points correspond to the trailing and leading edges,