Numerical Simulation of Turbulence

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A bit on what I do...

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What I do

Some turbulence concepts
Numerical simulation of Turbulence
Turbulence closure
LES of channel flow
A bit on what I do...
An eddy is a turbulent motion that occupies a region of size $l$ and has a characteristic velocity and timescale.
Reynolds Decomposition

\[ \phi(t) = \bar{\phi} + \phi'(t) \]
The transfer of energy from large eddies to successively smaller eddies is called the *energy cascade.*
Turbulent Energy Spectra

From Andre Bakker, www.bakker.org
Numerical techniques for simulating turbulence

• **Reynolds-averaged Navier-Stokes (RANS)**
  - Provides mean quantities of turbulent flows while all fluctuations are modeled
  - Computationally inexpensive

• **Large-eddy Simulation (LES)**
  - Resolves large-scale eddies while modeling small-scales
  - True LES resolves 80% of energy content
  - Requires significantly more resolution than RANS, particularly near walls

• **Direct Numerical Simulation (DNS)**
  - Simulates all scales of a turbulent flow
  - Extremely expensive: \( N_{3D} \sim Re_L^{9/4} \)
What each technique resolves

A sketch of scales resolved by different simulation techniques. Positions are approximate.
RANS example

From www.discretizer.org
LES and DNS example

From Wikipedia, Large-eddy Simulation
DNS Governing Equations

**Continuity Equation**

\[ \frac{\partial u_i}{\partial x_i} = 0 \]

**Momentum Equations**

\[ \frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_j} + 2 \frac{\partial}{\partial x_j} \nu S_{ij} \]

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
LES Governing Equations

Filtered Continuity Equation

$$\frac{\partial \hat{u}_i}{\partial x_i} = 0$$

Filtered Momentum Equations

$$\frac{\partial \hat{u}_i}{\partial t} + \frac{\partial (\hat{u}_i \hat{u}_j)}{\partial x_j} = - \frac{1}{\rho} \frac{\partial p}{\partial x_j} + 2 \frac{\partial}{\partial x_j} \nu S_{ij} - \frac{\partial \tau_{ij}}{\partial x_j}$$

$$S_{ij} = \frac{1}{2} \left( \frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i} \right)$$

$$\tau_{ij} = \hat{u}_i \hat{u}_j - \hat{u}_i \hat{u}_j$$
RANS Governing Equations

**Reynolds-Averaged Continuity Equation**

\[
\frac{\partial \langle u_i \rangle}{\partial x_i} = 0
\]

**Reynolds-Averaged Momentum Equations**

\[
\frac{\partial \langle u_i \rangle}{\partial t} + \frac{\partial (\langle u_i \rangle \langle u_j \rangle)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} + 2 \frac{\partial}{\partial x_j} \nu S_{ij} - \frac{\partial \tau_{ij}}{\partial x_j}
\]

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)
\]

\[
\tau_{ij} = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle
\]
Closing the RANS or LES equations

- For RANS, one can replace $\tau_{ij}$ with a turbulence model account for all of the fluctuations.
- Under the assumption that small-scales below the filter width are statistically universal, one can replace $\tau_{ij}$ with a sub-grid scale (SGS) model.
- The most common type of model (for both RANS and LES) is the eddy viscosity model:

$$\tau_{ij} = -2\nu_t S_{ij}$$

where the last two terms in momentum equation become

$$2 \frac{\partial}{\partial x_j} \nu S_{ij} - \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} (\nu + \nu_t) \left( \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right)$$
Common RANS eddy viscosity turbulence models

How to find the eddy viscosity, $\nu_t$, in RANS:

- Zero-equation: mixing length model ($\nu_t = (\kappa y)^2 |S|$)
- One-equation: Spalart-Allmaras (an equation for $k$)
- Two-equation: $k$-$\varepsilon$, RNG $k$-$\varepsilon$, realizable $k$-$\varepsilon$, $k$-$\omega$, algebraic stress model (ASM)
- Seven-equation: Reynolds stress model (RSM)

Note that:

- Equations refer to number of PDEs that require solving.
- $k$ stands for turbulent kinetic energy (TKE), or kinetic energy of only the fluctuations.
- $k$-$\varepsilon$ is the most common RANS turbulence model.
Common LES sub-grid (SGS) scale models

Smagorinsky Model

\[ \tau_{ij} = -2 \nu_t S_{ij} = -2 \left[ (C_S \Delta)^2 |S| \right] S_{ij} \]

- The SGS shear stress is proportional to the strain rate.
- Compare to Prandtl mixing length model: \( \nu_t = (\kappa y)^2 |S| \)
- \( \Delta \) represents the filter width.
- \( C_S \) is often calculated dynamically using flow field information (as opposed to van Driest damping).
  - Procedure often involves multiple filters using similarity principle in inertial subrange.

Other one- and two-equation SGS models exist as well. As in RANS, the PDEs are for TKE and dissipation.
Channel Setup

- \( \text{Re}_\tau = \frac{u_\tau h}{\nu} = 395 \) (\( \text{Re}_U \approx 13,000 \))
- \( \Delta z^+ = \frac{u_\tau \Delta z}{\nu} \approx 1 \), \( \Delta z \) is grid spacing in wall-normal direction
- 9.5 million cells
- Lagrangian dynamic Smagorinsky model

Channel Setup

\begin{align*}
\text{Re}_\tau &= \frac{u_\tau h}{\nu} = 395 \quad (\text{Re}_U \approx 13,000) \\
\Delta z^+ &= \frac{u_\tau \Delta z}{\nu} \approx 1, \, \Delta z \quad \text{is grid spacing in wall-normal direction} \\
&\text{9.5 million cells} \\
&\text{Lagrangian dynamic Smagorinsky model}
\end{align*}
Q-Criterion Visualization

LES of a Turbulent Channel Flow
Re$_T$ = 395 Mean Velocity Profile

DNS: Moser, Kim and Mansour, 1999
$Re_T = 395$ Reynolds Stress

DNS: Moser, Kim and Mansour, 1999
Movie Time!

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