10.1

Shear-Stress Distribution Across a Pipe Section

The velocity distribution in a pipe is directly linked to the shear-stress distribution; hence it is important to understand the latter. To determine the shear-stress distribution, we start with the equation of equilibrium applied to a cylindrical control volume that is oriented coaxially with the pipe, as shown in Fig. 10.1. For the conditions shown in Fig. 10.1, it is assumed that the flow is uniform (streamlines are straight and parallel). Therefore, the net momentum flow through the control volume is zero. Also, the pressure across any section of the pipe will be hydrostatically distributed. Thus the pressure force acting on an end face of the fluid element will be the product of the pressure at the center of the element (also at the center of the pipe) and the area of the face of the element. With steady uniform flow, equilibrium between the pressure, gravity, and shearing forces acting on the fluid will prevail. Consequently, the momentum equation yields the following:

$$\sum F_s = 0$$

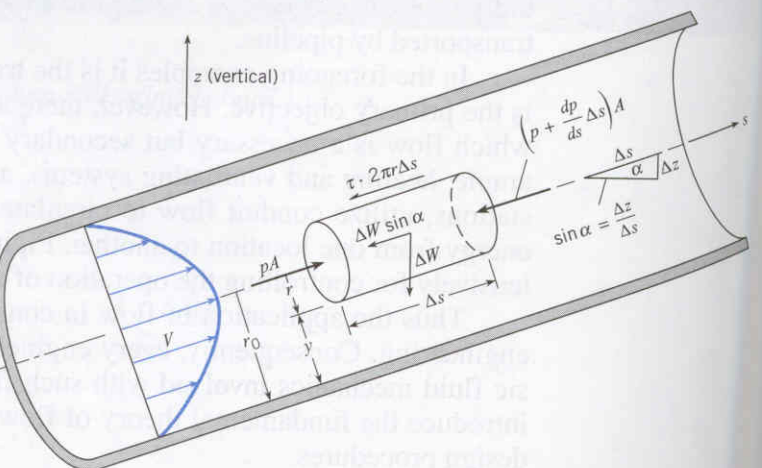
$$pA - \left(p + \frac{dp}{ds}\Delta s\right)A - \Delta W \sin\alpha - \tau(2\pi r)\Delta s = 0 \qquad (10.1)$$

In Eq. (10.1) $\Delta W = \gamma A \Delta s$ and $\sin\alpha = dz/ds$. Therefore, Eq. (10.1) reduces to

$$-\frac{dp}{ds}\Delta s A - \gamma A \Delta s \frac{dz}{ds} - \tau(2\pi r)\Delta s = 0 \qquad (10.2)$$

FIGURE 10.1

Flow in a pipe.



Then, when we divide Eq. (10.2) through by ΔsA and simplify, we obtain

$$\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.3}$$

Since the gradient itself, $d/ds\,(p + \gamma z)$, is negative (see Sec. 7.4) and constant across the section for uniform flow,* it follows that $-d/ds(p + \gamma z)$ will be positive and constant across the pipe section. Thus τ in Eq. (10.3) will be zero at the center of the pipe and will increase linearly to a maximum at the pipe wall. We will use Eq. (10.3) in the following section to derive the velocity distribution for laminar flow.

10.2

Laminar Flow in Pipes

We determine how the velocity varies across the pipe by substituting for τ in Eq. (10.3) its equivalent $\mu\, dV/dy$ and integrating. First, making the substitution, we have

$$\mu\frac{dV}{dy} = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.4}$$

Because $dV/dy = -dV/dr$, Eq. (10.4) becomes

$$\frac{dV}{dr} = -\frac{r}{2\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.5}$$

When we separate variables and integrate across the section, we obtain

$$V = -\frac{r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] + C \tag{10.6}$$

We can evaluate the constant of integration in Eq. (10.6) by noting that when $r = r_0$, the velocity $V = 0$. Therefore, the constant of integration is given by $C = (r_0^2/4\mu)[-d/ds(p + \gamma z)]$, and Eq. (10.6) then becomes

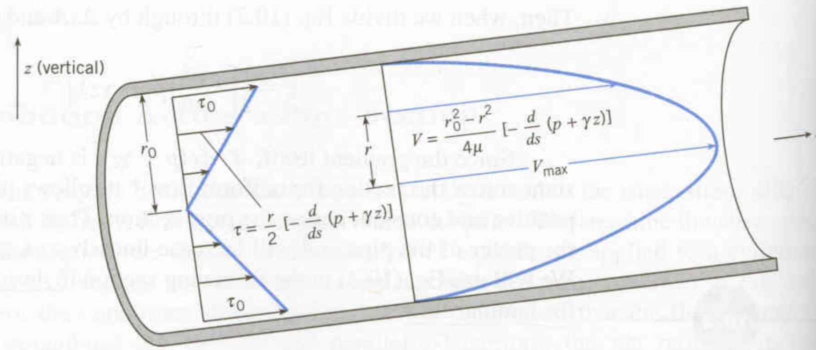
$$V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.7}$$

Equation (10.7) indicates that the velocity distribution for laminar flow in a pipe is parabolic across the section with the maximum velocity at the center of the pipe. Figure 10.2 shows the variation of the shear stress and velocity in the pipe.

* The combination $p + \gamma z$ is constant across the section because the streamlines are straight and parallel in uniform flow, and for this condition there will be no acceleration of the fluid normal to the streamline. Thus hydrostatic conditions prevail across the flow section. For a hydrostatic condition, $p/\gamma + z =$ constant or $p + \gamma z =$ constant as shown in Chapter 3.

FIGURE 10.2

Distribution of shear stress and velocity for laminar flow in a pipe.



Laminar flow in a round pipe is known as *Hagen-Poiseuille flow* named after a German, Hagen, and a Frenchman, Poiseuille, who studied low-speed flows in tubes in the 1840s.

example 10.1

Oil (S = 0.90; $\mu = 5 \times 10^{-1}$ N · s/m^2) flows steadily in a 3-cm pipe. The pipe is vertical, and the pressure at an elevation of 100 m is 200 kPa. If the pressure at an elevation of 85 m is 250 kPa, is the flow direction up or down? What is the velocity at the center of the pipe and at 6 mm from the center, assuming that the flow is laminar?

Solution First determine the rate of change of $p + \gamma z$. Taking s in the z direction,

$$\frac{d}{ds}(p + \gamma z) = \frac{(p_{100} + \gamma z_{100}) - (p_{85} + \gamma z_{85})}{15}$$

$$= \frac{[200 \times 10^3 + 8830(100)] - [250 \times 10^3 + 8830(85)]}{15}$$

$$= \frac{(1.083 \times 10^6 - 1.00 \times 10^6)\ \text{N/m}^2}{15\ \text{m}} = 5.53\ \text{kN/m}^3$$

The quantity $p + \gamma z$ is not constant with elevation—it increases upward (decreases downward). Therefore, the direction of flow is downward. This can be seen by substituting $d(p + \gamma z)/ds = 5.53$ kN/m^3 into Eq. (10.7). When this is done, V is negative for all values of r in the flow. When $r = 0$ (center of the pipe), the velocity will be maximum. Thus

$$V_{center} = V_{max} = \frac{r_0^2}{4\mu}(-5.53 \text{ kN/m}^3)$$

$$= \frac{0.015^2 \text{ m}^2}{4(5 \times 10^{-1} \text{ N} \cdot \text{s/m}^2)}(-5.53 \times 10^3 \text{ N/m}^3) = -0.622 \text{ m/s} \quad \triangleleft$$

At first it may seem strange that the velocity is in a direction opposite to the direction of decreasing pressure. However, it may not seem so peculiar if one realizes that in this example the pipe is vertical, so the gravitational force as well as pressure helps to establish the flow. What counts when flow is other than in the horizontal direction is how the combination $p + \gamma z$ changes with s. If $p + \gamma z$ is constant, then we have the equation of hydrostatics and no flow occurs. However, if $p + \gamma z$ is not constant, flow will occur in the direction of decreasing $p + \gamma z$.

Next determine the velocity at $r = 6$ mm $= 0.006$ m. Using Eq. (10.7), we find that

$$V = \frac{0.015^2 \text{ m}^2 - 0.006^2 \text{ m}^2}{4(5 \times 10^{-1} \text{ N} \cdot \text{s/m}^2)}(-5.53 \times 10^3 \text{ N/m}^3) = -0.522 \text{ m/s} \quad \triangleleft$$

For many problems we wish to relate the pressure change to the rate of flow or mean velocity $\bar{V}$ in the conduit. Therefore, it is necessary to integrate $dQ = V dA$ over the cross-sectional area of flow. That is,

$$Q = \int V \, dA$$
$$= \int_0^{r_0} \frac{(r_0^2 - r^2)}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right](2\pi r \, dr) \tag{10.8}$$

The factor $\pi[d(p + \gamma z)/ds]/4\mu$ is constant across the pipe section. Therefore, upon integration, we obtain

$$Q = \frac{\pi}{4\mu}\left[\frac{d}{ds}(p + \gamma z)\right]\frac{(r^2 - r_0^2)^2}{2}\Bigg|_0^{r_0} \tag{10.9}$$

which reduces to

$$Q = \frac{\pi r_0^4}{8\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.10}$$

If we divide through by the cross-sectional area of the pipe, we have an expression for the mean velocity:

$$\overline{V} = \frac{r_0^2}{8\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.11}$$

Comparing Eqs. (10.11) and (10.7) reveals that $\overline{V} = V_{max}/2$. Also, by substituting $D/2$ for r_0, we have

$$\overline{V} = \frac{D^2}{32\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{10.12}$$

or

$$\frac{d}{ds}(p + \gamma z) = -\frac{32\mu\overline{V}}{D^2} \tag{10.13}$$

Integrating Eq. (10.13) along the pipe between sections 1 and 2, we obtain

here is where the length L comes in.

$$p_2 - p_1 + \gamma(z_2 - z_1) = -\frac{32\mu\overline{V}}{D^2}(s_2 - s_1) \tag{10.14}$$

Here $s_2 - s_1$ is the length L of pipe between the two sections. Therefore, Eq. (10.14) can be rewritten as

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + \frac{32\mu L\overline{V}}{\gamma D^2} \tag{10.15}$$

It can be seen that when the general energy equation for incompressible flow in conduits, Eq. (7.24), is reduced to one for uniform flow in a constant-diameter pipe where $V_1 = V_2$, the result is

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + h_f \tag{10.16}$$

Here h_f is used instead of h_L to signify head loss due to frictional resistance of the pipe. Comparison of Eqs. (10.15) and (10.16) then shows that the head loss is given by

$$h_f = \frac{32\mu LV}{\gamma D^2} \tag{10.17}$$

Here the bar over the V has been omitted to conform to the standard practice of denoting the mean velocity in one-dimensional flow analyses by V without the bar.

10.3