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Frame Properties of Low Autocorrelation Random Sequences

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Abstract. The goal is to construct random frames and study properties of such frames. Starting with the construction of unimodular random sequences whose expected autocorrelations can be made arbitrarily low outside the origin, these random sequences are used to construct frames for $\mathbb{C}^d$. Using recent theory of non-asymptotic analysis of random matrices, the eigenvalue distribution of the corresponding frame operator is studied.

Keywords. Autocorrelation, Frames, Random matrix, Sequences.

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1 Introduction

1.1 Motivation

Let $\mathbb{R}$ be the real numbers, $\mathbb{Z}$ the integers, $\mathbb{C}$ the complex numbers, and set $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A fundamental problem in harmonic analysis is to characterize the family of positive bounded Radon measures F, whose inverse Fourier transforms are the autocorrelations of bounded sequences X. A special case of this is when $F \equiv 1$ on $\mathbb{T}$. Then the autocorrelation of X vanishes everywhere except at the origin, where it is equal to 1. The construction of unimodular sequences [1] with such spike like autocorrelation finds a lot of applications and is particularly relevant in the areas of radar and communications. In the former, the sequences can play a role in effective target recognition, e.g., [1,4,8,10–12,15,16]; and in the latter they are used to address synchronization issues in cellular (phone) access technologies, especially code division multiple access (CDMA), e.g., [17,18]. There are two main reasons that the sequences should be unimodular, that is, have constant amplitude. First, a transmitter can operate at peak power if the signal has constant peak amplitude - the system does not have to deal with the surprise of greater than expected amplitudes. Second, amplitude variations during transmission due to additive noise can be theoretically eliminated. The zero autocorrelation property ensures minimum

[1] Often in the literature, the term waveform is used instead of sequence

interference between signals sharing the same channel or between a signal and its reflection as might be needed in radar target recognition.

Deterministic construction of unimodular sequences with an impulse-like autocorrelation has been extensively studied in [3]. Here the focus is on random unimodular sequences with a probabilistic nature of construction. This might have an added advantage in applications by making such sequences harder to intercept by an adversary. The expectation of the autocorrelation is studied and it is desired that the expected autocorrelation is small everywhere outside the origin. Previous work on the use of stochastic sequences in radar can be found in [14], [13], [2] where the sequences or waveforms are only real-valued and not unimodular. In comparison, the sequences constructed here are complex valued and unimodular.

Frames are mathematical objects that can be thought of as redundant bases. In fact, for a finite dimensional vector space, a finite frame is the same as a spanning set. Frames have been widely popularized over the past two decades as standard tools in signal processing. Frames offer robust signal representations that are resilient to noise and transmission losses. Another aspect of the work presented here is to construct frames for $\mathbb{C}^d$ from low autocorrelation random sequences and to study the frame properties of such frames.

1.2 Notation and preliminaries

The *aperiodic autocorrelation* $A_X : \mathbb{Z} \to \mathbb{C}$ of a sequence $X : \mathbb{Z} \to \mathbb{C}$ is defined as

$$\forall k \in \mathbb{Z}, \quad A_X(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} X(m+k)\overline{X(m)} \tag{1.1}$$

provided the limit exists. Instead of aperiodic sequences that are defined on $\mathbb{Z}$, in some applications, it might be useful to construct periodic sequences with similar vanishing properties of the autocorrelation function. Let $n \geq 1$ be an integer and $\mathbb{Z}_n$ be the finite group $\{0, 1, \ldots, n-1\}$ with addition modulo n. The periodic autocorrelation $A_X : \mathbb{Z}_n \to \mathbb{C}$ of a sequence $X : \mathbb{Z}_n \to \mathbb{C}$ is defined as

$$\forall k = 0, 1, \ldots, n-1, \quad A_X(k) = \frac{1}{n} \sum_{m=0}^{n-1} X(m+k)\overline{X(m)}. \tag{1.2}$$

It is said that $X : \mathbb{Z}_n \to \mathbb{C}$ is a constant amplitude zero autocorrelation (CAZAC) sequence if each $|X(k)| = 1$ and

$$\forall k = 1, \ldots, n-1, \quad A_X(k) = \frac{1}{n} \sum_{m=0}^{n-1} X(m+k)\overline{X(m)} = 0.$$

The literature on CAZACs is overwhelming. A good reference on this topic is [8], among many others. Comparison between periodic and aperiodic autocorrelation can be found in [5]. Previously, a mathematical characterization of CAZACs in terms of finite unit-normed tight frames (FUNTFs) has been done in [4]. Here the focus is on infinite sequences and hence aperiodic autocorrelation.

Let X be a random variable with probability density function f. Assuming X to be absolutely continuous, the *expectation* of X, denoted by $E(X)$, is

$$E(X) = \int_{\mathbb{R}} x f(x)\,\mathrm{d}x.$$

The Gaussian random variable has probability density function given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$. The mean or expectation of this random variable is μ and the variance, $V(X)$, is σ^2. In this case it is also said that X follows a normal distribution and is written as $X \sim N(\mu, \sigma^2)$. The *characteristic function* of X at t, $E(e^{itX})$, is denoted by $\phi_X(t)$. For further properties of expectation and characteristic function of a random variable the reader is referred to [9].

Let $\mathcal{H}$ be a Hilbert space and let $V = \{v_k,\ k \in \mathcal{K}\}$, where $\mathcal{K}$ is some index set, be a collection of vectors in $\mathcal{H}$. Then V is said to be a *frame* for $\mathcal{H}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that for any $v \in \mathcal{H}$

$$A\|v\|^2 \leq \sum_{k \in \mathcal{K}} |\langle v, v_k \rangle|^2 \leq B\|v\|^2.$$

The constants A and B are called the *frame bounds*. The *lower frame bound* is A while the *upper frame bound* is B. If $A = B$, the frame is said to be *tight*. If $A = B = 1$, then the frame is called a *Parseval frame*. Orthonormal bases are special cases of Parseval frames.

If V is a frame for $\mathcal{H}$ then the map $F : \mathcal{H} \to \ell_2(\mathcal{K})$ given by $F(v) = \{\langle v, v_k \rangle : k \in \mathcal{K}\}$ is called the *analysis operator.* The *synthesis operator* is the adjoint map $F^* : \ell_2(\mathcal{K}) \to \mathcal{H}$, given by

$$F^*(\{a_k\}) = \sum_{k \in \mathcal{K}} a_k v_k.$$

The *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is given by $S = F^*F$. For a tight frame, the frame operator is just a constant multiple of the identity, i.e., $S = A\mathcal{I}$, where $\mathcal{I}$ is the identity map. Every $v \in \mathcal{H}$ can be represented as

$$v = \sum_{k \in \mathcal{K}} \langle v, S^{-1}v_k \rangle v_k = \sum_{k \in \mathcal{K}} \langle v, v_k \rangle S^{-1} v_k.$$

Here $\{S^{-1}v_k\}$ is also a frame and is called the *dual frame*. For a tight frame, S^{-1} is just $\frac{1}{A}S$. Tight frames are thus highly desirable since they offer a computationally simple reconstruction formula that does not involve inverting the frame operator. The minimum and maximum eigenvalues of S are the optimal lower and upper frame bounds respectively [6]. Thus, for a tight frame all the eigenvalues of the frame operator are equal to each other. For the general theory on frames one can refer to [6], [7].

For a random vector $X \in \mathbb{C}^n$, the second moment matrix, which takes the place of second moment of a random variable, is defined as

$$\Sigma = E(X \otimes X) = E(XX^*)$$

where $\otimes$ denotes outer product. A random vector is called *isotropic* if $\Sigma(X) = \mathcal{I}$. The following results [19] will be used to determine frame bounds for random frames. The smallest and largest singular values of a matrix F are denoted by $s_{\min}(F)$ and $s_{\max}(F)$, respectively. The probability of an event E is denoted by $P(E)$.

Lemma 1.1. [19] *If for some $\delta > 0$, a matrix F satisfies*

$$\|F^*F - \mathcal{I}\| \leq \max(\delta, \delta^2)$$

then

$$1 - \delta \leq s_{\min}(F) \leq s_{\max}(F) \leq 1 + \delta.$$

Theorem 1.2. [19] *Let F be an $M \times d$ matrix whose rows v_k are independent random vectors in $\mathbb{R}^d$ with common second moment matrix $\Sigma = E(v_k \otimes v_k)$. For each k, suppose $\|v_k\| \leq m$ almost surely, for some constant m. Then for every $t \geq 0$,*

$$P\left(\left\|\frac{1}{M}F^*F - \Sigma\right\| \leq \max\left(\|\Sigma\|^{1/2}\delta, \delta^2\right)\right) \geq 1 - de^{-ct^2}$$

where $\delta = t\sqrt{\frac{m}{M}}$ and $c > 0$ is an absolute constant.

1.3 Outline

In Section 2, random sequences $X : \mathbb{Z} \to \mathbb{C}$ are constructed such that the expected autocorrelation of X can be made arbitrarily small outside the origin, and for all $n \in \mathbb{Z}$, $|X[n]| = 1$. The higher dimensional case of constructing $X : \mathbb{Z} \to \mathbb{C}^d$ is also addressed. Section 3 discusses the construction of frames from low autocorrelation random sequences. Using random matrix theory, the frame properties of these random frames are also studied in Section 3. Section 4 includes some comments on the variance of the autocorrelation of the random sequences constructed in Section 2, as well as on the applicability of random frames in signal transmission.

2 Low autocorrelation unimodular random sequences

In this section unimodular sequences, $X : \mathbb{Z} \to \mathbb{C}$, are constructed from random variables such that the expectation of the autocorrelation can be made arbitrarily small everywhere except at the origin. First, such a construction is done using the Gaussian random variable. Next, a general characterization of all random variables that can be used for the purpose is given.

Let $\{Y_\ell\}_{\ell\in\mathbb{Z}}$ be independent identically distributed (i.i.d.) random variables such that $Y_\ell \sim N(0, \sigma^2)$. Define $X : \mathbb{Z} \to \mathbb{C}$ by

$$\forall n \in \mathbb{Z}, \quad X(n) = e^{\frac{2\pi i}{\epsilon} Y_n} \tag{2.1}$$

where i is $\sqrt{-1}$ and ϵ is a constant to be chosen. For each n, $|X(n)| = 1$ and thus X is unimodular. As introduced in Section 1.2, the autocorrelation of X at $k \in \mathbb{Z}$ is

$$A_X(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} X(n+k)\overline{X(n)}$$

where the limit is in the sense of probability.

Theorem 2.1. *Given $\epsilon > 0$, the waveform $X : \mathbb{Z} \to \mathbb{C}$ defined in (2.1) has autocorrelation A_X such that*

$$E(A_X(k)) = \begin{cases} 1 & \text{if } k = 0 \\ e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} & \text{if } k \neq 0. \end{cases}$$

Proof. (i) When $k = 0$,

$$A_X(0) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} X(n)\overline{X(n)} = 1.$$

and so $E(A_X(0)) = 1$.
(ii) Let $k \neq 0$. One would like to calculate

$$E(A_X(k)) = E\left(\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} X(n+k)\overline{X(n)}\right).$$

Let $g_N(X) = \frac{1}{2N+1}\sum_{n=-N}^{N} X(n+k)\overline{X(n)}$. Then $|g_N(X)| \leq 1$. Let $h(X) = 1$. Then for each N, $|g_N(X)| \leq h(X)$ and $E(h(X)) = 1$. Thus, by the Dominated Convergence Theorem [9], which justifies the interchange of limit and integration below, one obtains

$$\begin{aligned}
E(A_X(k)) &= E\left(\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} X(n+k)\overline{X(n)}\right) \\
&= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} E(X(n+k)\overline{X(n)}) \\
&= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} E(e^{\frac{2\pi i}{\epsilon}(Y_{n+k}-Y_n)}) \\
&= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} E\left(e^{\frac{2\pi i}{\epsilon}Y_{n+k}}\right) E\left(e^{-\frac{2\pi i}{\epsilon}Y_n}\right) \\
&= E\left(e^{\frac{2\pi i}{\epsilon}Y_1}\right) E\left(e^{-\frac{2\pi i}{\epsilon}Y_1}\right) = \left[\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right)\right]^2
\end{aligned}$$

where the last line uses the fact that the Y_ℓs are i.i.d. random variables. Here $\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right)$ is the characteristic function at $\frac{2\pi}{\epsilon}$ of Y_1 which is the same as that of any other Y_ℓ due to their identical distribution. Since the characteristic function of a Gaussian random variable is even, $\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right) = \phi_{Y_1}\left(-\frac{2\pi}{\epsilon}\right)$. The characteristic function at $\frac{2\pi}{\epsilon}$ of a Gaussian random variable with mean 0 and variance σ^2 is $e^{-\frac{\sigma^2}{2}\left(\frac{2\pi}{\epsilon}\right)^2}$. Thus

$$E(A_X(k)) = e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}.$$

□

As can be seen, the autocorrelation can be made arbitrarily small for $k \neq 0$ by choosing ϵ to be small. This construction can be generalized to other random variables.

Theorem 2.2. *Let $\{Y_\ell\}_{\ell\in\mathbb{Z}}$ be a sequence of i.i.d. random variables with characteristic function ϕ_Y. Suppose that the probability density function of the Y_ℓs is even and that $\phi_Y(t)$ goes to 0 as t goes to infinity. Then, given ϵ, the sequence $X : \mathbb{Z} \to \mathbb{C}$ given by*

$$X(n) = e^{\frac{2\pi}{\epsilon}iY_n}$$

has autocorrelation that can be made arbitrarily small outside the origin and is given by

$$E(A_X(k)) = \begin{cases} 1 & \textit{when } k = 0 \\ \left[\phi_Y\left(\frac{2\pi}{\epsilon}\right)\right]^2 & \textit{when } k \neq 0. \end{cases}$$

Proof. By unimodularity of the sequence, $E(A_X(0)) = 1$. Since the density function of each Y_ℓ is even this means that the characteristic function is real valued [9]. Following the calculation in the proof of Theorem 2.1, the expected autocorrelation of X for $k \neq 0$ is

$$E(A_X(k)) = \left[\phi_Y\left(\frac{2\pi}{\epsilon}\right)\right]^2$$

and this goes to zero with ϵ by the hypothesis. □

Example 2.3. Suppose the Y_ℓs follow a *bilateral* distribution that has density $e^{-|x|}$ with $x \in (-\infty, \infty)$ and characteristic function $\phi_Y(t) = \frac{1}{1+t^2}$. Then for $k \neq 0$,

$$E(A_X(k)) = \left[\frac{1}{1 + \left(\frac{2\pi}{\epsilon}\right)^2}\right]^2$$

and this can be made arbitrarily small with ϵ.

Example 2.4. Suppose that the Y_ℓs follow the *Cauchy* distribution with density function $\frac{1}{\pi(1+x^2)}$. Note that disregarding the constant π, this is the characteristic function of the random variable considered in Example 2.3. The characteristic function of the Y_ℓs is now $e^{-|t|}$, the same as the distribution function in Example 2.3. For $k \neq 0$,

$$E(A_X(k)) = \left[\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right)\right]^2 = e^{-\frac{4\pi}{\epsilon}}$$

which can be made arbitrarily small with ϵ.

For the Gaussian case, the variance of the autocorrelation is calculated in Section 4.

Higher dimensional case: Here one is interested in constructing sequences $v : \mathbb{Z} \to \mathbb{C}^d$, $d \geq 2$. It is desired that v has unit norm and the expectation of its autocorrelation can be made arbitrarily small. Define a sequence X in $\mathbb{C}$ as

$X(n) = e^{\frac{2\pi}{\epsilon} iY_n}$, $Y_n \sim N(0, \sigma^2)$. The following is motivated by the higher dimensional treatment in the deterministic case [4]. A sequence $v : \mathbb{Z} \to \mathbb{C}^d$ is then defined as

$$\forall m \in \mathbb{Z}, \quad v(m) = \frac{1}{\sqrt{d}} \begin{bmatrix} X(m) \\ X(m+1) \\ \vdots \\ X(m+d-1) \end{bmatrix}. \tag{2.2}$$

In this case, the autocorrelation is given by

$$A_v(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \langle v(n+k), v(n) \rangle \tag{2.3}$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{C}^d$. The norm of any $v(m)$ is thus given by

$$\|v(m)\|^2 = \langle v(m), v(m) \rangle.$$

From (2.2),

$$\|v(m)\|^2 \quad = \quad \frac{1}{d} \sum_{n=0}^{d-1} X(m+n)\overline{X(m+n)} = \frac{d}{d} = 1.$$

Thus the $v(m)$s are unit-normed. The following Theorem 2.5 shows that the expected autocorrelation of v can be made arbitrarily small everywhere except at the origin.

Theorem 2.5. *Given $\epsilon > 0$, the sequence $v : \mathbb{Z} \to \mathbb{C}^d$ defined in (2.2) has autocorrelation A_v such that*

$$E(A_v(k)) = \begin{cases} 1 & \text{if } k = 0 \\ e^{-\sigma^2 \left(\frac{2\pi}{\epsilon}\right)^2} & \text{if } k \neq 0. \end{cases}$$

Proof. When $k = 0$,

$$A_v(0) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \|v(n)\|^2 = 1.$$

Thus,

$$E(A_v(0)) = 1.$$

Consider $k \neq 0$.

$$\begin{aligned}
E(A_v(k)) &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} E(\langle v(n+k), v(n)\rangle) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} E\left(\frac{1}{d}\sum_{m=0}^{d-1} X(n+k+m)\overline{X(n+m)}\right) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{d}\sum_{m=0}^{d-1} E\left(X(n+k+m)\overline{X(n+m)}\right) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{d}\sum_{m=0}^{d-1} E\left(e^{\frac{2\pi}{\epsilon} i(Y_{n+m+k}-Y_{n+m})}\right) \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1}{d}\sum_{m=0}^{d-1} \left[\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right)\right]^2 = e^{-\sigma^2(\frac{2\pi}{\epsilon})^2}
\end{aligned}$$

where the last step uses the fact that the Y_ℓs are i.i.d. random variables and the characteristic function of the Gaussian. □

Remark 2.6. As in the one dimensional case, one can see that here too the construction can be done with random variables other than the Gaussian. In fact, all random variables that can be used in the one dimensional case, i.e., ones satisfying the properties of Theorem 2.2, can also be used for the higher dimensional construction.

Remark 2.7 (Remark on the periodic case). It can be shown that the periodic case follows the same nature as the aperiodic case. The sequence $X : \mathbb{Z}_n \to \mathbb{C}$ is defined in the same way as in the start of Section 2, i.e.,

$$\forall m \in \{0, 1, \ldots, n-1\}, \quad X(m) = e^{\frac{2\pi}{\epsilon} iY_m}$$

where $Y_m \sim N(0, \sigma^2)$. Following the definition given in (1.2), when $k = 0$,

$$A_X(0) = \frac{1}{n}\sum_{m=0}^{n-1} X(m)\overline{X(m)} = 1.$$

When $k \neq 0$, the expectation of the autocorrelation is

$$E(A_X(k)) = \frac{1}{n}\sum_{m=0}^{n-1} E(X(m+k)\overline{X(m)})$$

which can be calculated as

$$
\begin{aligned}
E(A_X(k)) &= \frac{1}{n}\sum_{m=0}^{n-1} E(e^{\frac{2\pi i}{\epsilon}(Y_{m+k}-Y_m)}) = \frac{1}{n}\sum_{m=0}^{n-1}\left[E\left(e^{\frac{2\pi i}{\epsilon}Y_1}\right)\right]^2 \\
&= \left[E\left(e^{\frac{2\pi i}{\epsilon}Y_1}\right)\right]^2 = \left[\phi_{Y_1}\left(\frac{2\pi}{\epsilon}\right)\right]^2 = e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}
\end{aligned}
$$

where one uses the fact that the Y_ms are i.i.d.. This suggests that the autocorrelation can be made arbitrarily small, depending on ϵ, for all non-zero values of k. Also, as in the aperiodic case, this result can be obtained for random variables other than the Gaussian.

3 Frames from low autocorrelation random sequences

Let $\{Y_{mn}\}_{m,n\in\mathbb{Z}}$ be i.i.d. random variables following a Gaussian distribution [2] with mean zero and variance σ^2, i.e., $Y_{mn} \sim N(0,\sigma^2)$. For a given ϵ,

$$E(e^{\frac{2\pi}{\epsilon}iY_{mn}}) = e^{-\frac{\sigma^2}{2}\left(\frac{2\pi}{\epsilon}\right)^2}$$

and the variance

$$V(e^{\frac{2\pi}{\epsilon}iY_{mn}}) = 1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}.$$

For $m, n \in \mathbb{Z}$, define

$$X_{mn} = e^{\frac{2\pi}{\epsilon}iY_{mn}} - e^{-\frac{\sigma^2}{2}\left(\frac{2\pi}{\epsilon}\right)^2}.$$

Then X_{mn} has mean zero and variance $\widehat{\sigma}^2 = 1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$. Consider the mapping $v : \mathbb{Z} \to \mathbb{C}^d$ given by

$$v(\ell) := v_\ell = \frac{1}{\sqrt{d}}\begin{pmatrix} X_{\ell 1} \\ X_{\ell 2} \\ \vdots \\ X_{\ell d}\end{pmatrix}. \tag{3.1}$$

[2] All work shown in this section can be carried out with random variables following other distributions such as those given in Examples 2.3 and 2.4.

Consider the set of M vectors $V = \{v_1, v_2, \ldots, v_M\}$ in $\mathbb{C}^d$. Assuming $M \geq d$, the analysis operator of the set V is

$$F = \frac{1}{\sqrt{d}} \begin{bmatrix} \overline{X}_{11} & \overline{X}_{12} & \cdots & \overline{X}_{1d} \\ \overline{X}_{21} & \overline{X}_{22} & \cdots & \overline{X}_{2d} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{X}_{M1} & \overline{X}_{M2} & \cdots & \overline{X}_{Md} \end{bmatrix} = \begin{bmatrix} - & \overline{v}_1 & - \\ - & \overline{v}_2 & - \\ & \vdots & \\ - & \overline{v}_M & - \end{bmatrix} \tag{3.2}$$

so that $S = F^*F$ is the frame operator. The matrix F has i.i.d. entries with mean zero and variance $\hat{\sigma}^2 = \frac{1}{d}(1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2})$. Thus, for each k,

$$\begin{aligned} \Sigma &= E(v_k \otimes v_k) = \frac{1}{d} E \begin{bmatrix} |X_{k1}|^2 & X_{k1}\overline{X}_{k2} & \cdots & X_{k1}\overline{X}_{kd} \\ X_{k2}\overline{X}_{k1} & |X_{k2}|^2 & \cdots & X_{k2}\overline{X}_{kd} \\ \vdots & \vdots & \cdots & \vdots \\ X_{kd}\overline{X}_{k1} & X_{kd}\overline{X}_{k2} & \cdots & |X_{kd}|^2 \end{bmatrix} \\ &= \frac{1}{d} \begin{bmatrix} V(X_{k1}) & 0 & \cdots & 0 \\ 0 & V(X_{k2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V(X_{kd}) \end{bmatrix} = \frac{1}{d}\left(1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}\right)\mathcal{I}. \end{aligned}$$

3.1 Frame tightness

In this subsection, it will be shown that by taking ϵ to be small, the singular values of the analysis operator F or the eigenvalues of the frame operator S can be made close to each other with a high probability. For each k, the kth row of F satisfies $\|v_k\| \leq m$ where $m = 1 + e^{-\frac{\sigma^2}{2}\left(\frac{2\pi}{\epsilon}\right)^2}$. The matrix F for the frame V satisfies the conditions of Theorem 1.2. To make things look less cumbersome, let $\rho = 1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$. Then the second moment matrix for each row of F is $\Sigma = \frac{\rho}{d}\mathcal{I}$. By Theorem 1.2, [3] for every $t \geq 0$, with probability at least $1 - de^{-ct^2}$

$$\left\|\frac{1}{M}F^*F - \frac{\rho}{d}\mathcal{I}\right\| \leq \max\left(\|\Sigma\|^{1/2}\delta, \delta^2\right) \tag{3.3}$$

[3] It should be noted that even though Theorem 1.2 is for matrices with real entries, it is applicable in the present since the entries of F can be parametrized by elements on $\mathbb{T}$. Further, the results on non-asymptotic analysis of random matrices as discussed in [19] are all valid for complex entries with different constants.

where $\delta = t\sqrt{\frac{m}{M}}$ and c is an absolute constant. Note that $\|\Sigma\| = \frac{\rho}{d}$. For any $t \geq 0$, let

$$\hat{\epsilon} = \max\left(t\sqrt{\frac{\rho m}{dM}}, \frac{t^2 m}{M}\right).$$

The event in (3.3) can be reduced as follows

$$\begin{aligned}
& \left\|\frac{1}{M}S - \frac{\rho}{d}\mathcal{I}\right\| \leq \hat{\epsilon} \\
\implies \quad & \left\|\frac{d}{\rho}\frac{1}{M}S - \mathcal{I}\right\| \leq \frac{d}{\rho}\hat{\epsilon} \\
\implies \quad & \left\|\frac{d}{\rho}\frac{1}{M}S - \mathcal{I}\right\| \leq \max(\hat{\delta}, \hat{\delta}^2); \quad \hat{\delta} = \frac{t\sqrt{dm}}{\sqrt{M\rho}} \\
\implies \quad & 1 - \hat{\delta} \leq s_{\min}\left(\sqrt{\frac{d}{\rho M}}F\right) \leq s_{\max}\left(\sqrt{\frac{d}{\rho M}}F\right) \leq 1 + \hat{\delta} \quad \text{(by Lemma 1.1)} \\
\implies \quad & (1 - \hat{\delta})^2 \leq \lambda_{\min}\left(\frac{d}{\rho M}S\right) \leq \lambda_{\max}\left(\frac{d}{\rho M}S\right) \leq (1 + \hat{\delta})^2 \\
\implies \quad & \frac{\rho M}{d}(1 - \hat{\delta})^2 \leq \lambda_{\min}(S) \leq \lambda_{\max}(S) \leq \frac{\rho M}{d}(1 + \hat{\delta})^2.
\end{aligned}$$

Since $\rho = 1 - e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$, for sufficiently small ϵ, $\rho \approx 1$, $m \approx 1$, and $\hat{\delta} \approx \frac{t\sqrt{d}}{\sqrt{M}}$. This leads to the following.

Theorem 3.1. *For ϵ sufficiently small, for every $t \geq 0$,*

$$P\left(\frac{M}{d}\left(1 - t\sqrt{\frac{d}{M}}\right)^2 \leq \lambda_{\min}(S) \leq \lambda_{\max}(S) \leq \frac{M}{d}\left(1 + t\sqrt{\frac{d}{M}}\right)^2\right) \geq 1 - de^{-ct^2}$$

where the norm of the frame vectors is bounded above by m and $c > 0$ is an absolute constant.

Theorem 3.1 gives a sense of how the lower and upper frame bounds of V are distributed. For a frame with high redundancy ($M \gg d$), for a suitable t, the eigenvalues of the frame operator S are all close to each other and close to $\frac{M}{d}$ with a high probability. Roughly speaking, due to (3.3), when $\rho \approx 1$, the norm of the difference between S and $\frac{M}{d}\mathcal{I}$ can be made small with a high probability.

3.2 Equiangularity of random frames

Keeping other notation same as before, let $X_{k\ell} = e^{\frac{2\pi i}{\epsilon} Y_{k\ell}}$, so that unlike Section 3.1, $X_{k\ell}$ is not centered. Define a vector $f_j \in \mathbb{C}^d$ as

$$f_j = \frac{1}{\sqrt{d}} \begin{bmatrix} X_{j1} \\ X_{j2} \\ \vdots \\ X_{jd} \end{bmatrix}.$$

Consider the set $\{f_1, \ldots, f_M\}$. This is a set of unit-normed random vectors in $\mathbb{C}^d$. The expectation of the inner products among such random vectors is established here. The inner product of unit-normed vectors is a measure of the anglular distance them. Note that $|X_{k\ell}| = 1$ for all k and ℓ. For integers $k \neq \ell$ and $i \neq j$,

$$E(X_{ki} X_{\ell j} \overline{X_{kj}}\, \overline{X_{\ell i}}) = \left[\phi\left(\frac{2\pi}{\epsilon}\right)\right]^4 = e^{-2\sigma^2 (2\pi/\epsilon)^2}. \tag{3.4}$$

Using (3.4),

$$\begin{aligned} &E(|\langle f_k, f_\ell\rangle|^2) = E(\langle f_k, f_\ell\rangle \overline{\langle f_k, f_\ell\rangle}) \\ &= \frac{1}{d^2} E\left(\sum_{i=1}^{d} |X_{ki}|^2 |X_{\ell i}|^2 + \sum_{i\neq j} X_{ki} \overline{X_{kj}}\, \overline{X_{\ell i}} X_{\ell j}\right) \\ &= \frac{1}{d^2}(d + d(d-1) e^{-2\sigma^2(2\pi/\epsilon)^2}) = \frac{1}{d} + \frac{d-1}{d} e^{-2\sigma^2(2\pi/\epsilon)^2}. \end{aligned} \tag{3.5}$$

When ϵ is small, $E(|\langle f_k, f_\ell\rangle|^2) \approx \frac{1}{d}$ for $k \neq \ell$. The variance of the cross correlation can be calculated as follows.

$$V(|\langle f_k, f_\ell\rangle|^2) = E(|\langle f_k, f_\ell\rangle|^4) - \left(E\left(|\langle f_k, f_\ell\rangle|^2\right)\right)^2.$$

From (3.5),

$$\left(E\left(|\langle f_k, f_\ell\rangle|^2\right)\right)^2 = \frac{1}{d^2} + \frac{2(d-1)}{d^2} e^{-2\sigma^2(2\pi/\epsilon)^2} + \frac{(d-1)^2}{d^2} e^{-4\sigma^2(2\pi/\epsilon)^2}.$$

$$
\begin{aligned}
E(|\langle f_k, f_\ell\rangle|^4) &= \frac{1}{d^4} E\left(\sum_{i=1}^{d} |X_{ki}|^2 |X_{\ell i}|^2 + \sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right)^2 \\
&= \frac{1}{d^4} E\left(d + \sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right)^2 \\
&= \frac{1}{d^2} + \frac{2}{d^3} E\left(\sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right) + \frac{1}{d^4} E\left(\sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right)^2 .
\end{aligned} \tag{3.6}
$$

Further,

$$
E\left(\sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right) = d(d-1)\left(\phi\left(\frac{2\pi}{\epsilon}\right)\right)^4 = d(d-1)e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}, \tag{3.7}
$$

$$
\begin{aligned}
\left(\sum_{i\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j}\right)^2 \quad = \quad & \sum_{i\neq j} X_{ki}^2 \overline{X_{kj}}^2\, \overline{X_{\ell i}}^2 X_{\ell j}^2 \\
& + 2\sum_{i\neq j}\sum_{p\neq i}\sum_{q\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j} X_{kp}\overline{X_{kq}}\,\overline{X_{\ell p}} X_{\ell q},
\end{aligned}
$$

$$
E\left(\sum_{i\neq j} X_{ki}^2 \overline{X_{kj}}^2\, \overline{X_{\ell i}}^2 X_{\ell j}^2\right) = d(d-1)\left(\phi\left(\frac{4\pi}{\epsilon}\right)\right)^4 = d(d-1)e^{-2\sigma^2\left(\frac{4\pi}{\epsilon}\right)^2}, \tag{3.8}
$$

and

$$
E\left(\sum_{i\neq j}\sum_{p\neq i}\sum_{q\neq j} X_{ki}\overline{X_{kj}}\,\overline{X_{\ell i}} X_{\ell j} X_{kp}\overline{X_{kq}}\,\overline{X_{\ell p}} X_{\ell q}\right) = d_0 \phi\left(\frac{2\pi}{\epsilon}\right)^8 = d_0 e^{-4\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} \tag{3.9}
$$

where $d_0 = d^4 - d^3 - d^2 - d$. Using (3.7), (3.8), and (3.9) in (3.6) gives

$$
E(|\langle f_k, f_\ell\rangle|^4) = \frac{1}{d^2} + \frac{2(d-1)}{d^2} e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} + \frac{d-1}{d^3} e^{-2\sigma^2\left(\frac{4\pi}{\epsilon}\right)^2} + \frac{2d_0}{d^4} e^{-4\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}
$$

and consequently the variance is given by

$$
V(|\langle f_k, f_\ell\rangle|^2) = \frac{d-1}{d^3} e^{-2\sigma^2\left(\frac{4\pi}{\epsilon}\right)^2} + \frac{d^3 - 3d - 2}{d^3} e^{-4\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}.
$$

4 Discussion

Variance of the autocorrelation: The variance of the autocorrelation is given by

$$V(A_X(k)) = E(|A_X(k)|^2) - |E(A_X(k))|^2.$$

Restricting all calculations to the Gaussian random variable case,

$$|E(A_X(k))|^2 = e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}.$$

$$\begin{aligned}
|A_X(k)|^2 &= A_X(k)\overline{A_X(k)} \\
&= \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n=-N}^{N} \sum_{m=-M}^{M} X(n+k)\overline{X(n)}\,\overline{X(m+k)}X(m) \\
&= \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n=-N}^{N} \sum_{m=-M}^{M} e^{\frac{2\pi}{\epsilon}i(Y_{n+k}-Y_n-Y_{m+k}+Y_m)} \\
&= \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n=m} 1 \\
&\qquad + \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n\neq m} e^{\frac{2\pi}{\epsilon}i(Y_{n+k}-Y_n-Y_{m+k}+Y_m)} \\
&= \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n\neq m} e^{\frac{2\pi}{\epsilon}i(Y_{n+k}-Y_n-Y_{m+k}+Y_m)}.
\end{aligned}$$

On taking the expectation, the terms in the summation on the right are either $e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$, $e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$, or $e^{-3\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}$. Therefore,

$$E(|A_x(k)|^2) \le \lim_{N,M\to\infty} \frac{1}{(2N+1)(2M+1)} \sum_{n\neq m} e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} \le e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2},$$

and

$$0 \le V(A_x(k)) \le e^{-\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} - e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2}.$$

When the receiver knows part of a random frame: As described in Section 3, consider a random frame $\{f_1, f_2, \ldots, f_M\} \subset \mathbb{C}^d$ without centering so that $\|f_i\|_2 = 1$ for all i. A signal $x \in \mathbb{C}^d$ is transmitted by sending the coefficients $\{\langle x, f_i\rangle\}_{i=1}^M$. Denoting the frame operator by S, x can be reconstructed as

$$x = \sum_{i=1}^{M} \langle x, f_i\rangle S^{-1} f_i.$$

Suppose the receiver has access to part of the frame and does not know the entire frame that was used to compute the coefficients $\{\langle x, f_i\rangle\}_{i=1}^M$. For instance, the receiver may not know the kth vector f_k. If the receiver knows how the random vectors were generated then f_k can be replaced by $\hat{f}_k$ where the realization of the random variable used will most likely be different from the one used by the transmitter. Let $\hat{S}$ denote the frame operator of $\{f_1, \ldots, f_{k-1}, \hat{f}_k, f_{k+1}, \ldots, f_M\}$. The receiver then reconstructs x as

$$\hat{x} = \sum_{i \neq k} \langle x, f_i\rangle \hat{S}^{-1} f_i + \langle x, f_k\rangle \hat{S}^{-1} \hat{f}_k$$

and

$$x - \hat{x} = \sum_{i \neq k} \langle x, f_i\rangle (\hat{S}^{-1} - S^{-1}) f_i + \langle x, f_k\rangle (\hat{S}^{-1}\hat{f}_k - S^{-1} f_k).$$

Using the triangle inequality and the fact that $\|f_i\|_2 = 1$ for all i,

$$\begin{aligned}
\|x - \hat{x}\|_2 &\leq \|\sum_{i \neq k} \langle x, f_i\rangle (\hat{S}^{-1} - S^{-1}) f_i\|_2 + \|\langle x, f_k\rangle (\hat{S}^{-1}\hat{f}_k - S^{-1} f_k)\|_2 \\
&\leq \sum_{i \neq k} |\langle x, f_i\rangle| \|\hat{S}^{-1} - S^{-1}\| \|f_i\|_2 + |\langle x, f_k\rangle| \left(\|\hat{S}^{-1}\| \|\hat{f}_k\|_2 + \|S^{-1}\| \|f_k\|_2\right) \\
&= \sum_{i \neq k} |\langle x, f_i\rangle| \|\hat{S}^{-1} - S^{-1}\| + |\langle x, f_k\rangle| \left(\|\hat{S}^{-1}\| + \|S^{-1}\|\right) \\
&\leq (\|\hat{S}^{-1}\| + \|S^{-1}\|) \sum_{i=1}^{M} |\langle x, f_i\rangle| \leq (\|\hat{S}^{-1}\| + \|S^{-1}\|) M \|x\|_2.
\end{aligned}$$

The relative error is therefore bounded by

$$\frac{\|x - \hat{x}\|_2}{\|x\|_2} \leq M(\|\hat{S}^{-1}\| + \|S^{-1}\|).$$

Note that both S^{-1} and $\hat{S}^{-1}$ have the same distribution of the frame bounds coming from the inverse of the frame bounds of the corresponding frame operator. One might be able to justify, following the result in Theorem 3.1, that with a high probability the frame bounds are close to $\frac{d}{M}$. Then with a high probability, the relative error is bounded above by twice the dimension. Further study on the role of random frames in signal recovery when the receiver has part of the frame or when there are lost coefficients form part of future work. In the case of traditional frames, if the receiver does not know some part of the frame then, depending on how much is not known, the receiver may not be able to recover the signal at all.

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