Equiangular frames and their duals

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Abstract Systems of m equiangular lines spanning $\mathbb{R}^d$ or $\mathbb{C}^d$ that satisfy the so called Welch bound have recently gained a lot of attention due to various applications in signal processing. Such sets are called *equiangular tight frames* (ETFs). One of the geometrically appealing aspects of an ETF is that any vector can be represented in terms of an ETF by using a dual frame that is also an equiangular set. However, for a given m and d, with $m > d+1$, ETFs are rare. Here we study some properties of equiangular lines spanning $\mathbb{R}^d$ when the Welch bound is not met. Such equiangular sets are more common than ETFs. In this case, the properties of the canonical dual, in particular, the angle set of the canonical dual are studied. We determine conditions on equiangular lines spanning $\mathbb{R}^d$ whose canonical dual has few distinct angles.
Keywords Equiangular frames, k-angle frames, Signature matrices, Welch bound

1 Introduction

Given a set $\{f_i\}_{i=1}^m$ of m unit vectors in $\mathbb{C}^d$, with $m > d$, the lower bound on the maximum cross correlation between distinct vectors is given by

$$\max_{i \neq j} |\langle f_i, f_j \rangle|^2 \geq \frac{m-d}{d(m-1)}. \tag{1}$$

The quantity on the right of (1) is known as the *Welch bound* after L. R. Welch who gave a family of bounds [29] parametrized by integers $k \geq 1$ as follows.

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$$\max_{i \neq j} |\langle f_i, f_j \rangle|^{2k} \geq \frac{1}{m-1} \left[\frac{m}{\binom{d+k-1}{k}} - 1 \right]. \tag{2}$$

Welch obtained the bounds in (2) as a consequence of the following inequality

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle f_i, f_j \rangle|^{2k} \geq \frac{m^2}{\binom{d+k-1}{k}},$$

and often this is referred to as the Welch bound. The special case of $k = 1$ which is given in (1) has gained a lot of attention among researchers mainly in regard to the study of sets that attain the lower bound. Sets that attain the lower bound in (1), often called *Welch bound equality* sets, arise in various application areas such as communication systems, quantum information processing, and coding theory [18, 21, 23, 30, 16, 19, 22, 20, 14]. In a purely mathematical setting, sets that attain the lower bound in (1) are objects called *equiangular tight frames* (ETFs) . Consequently, the problem of constructing ETFs and determining conditions under which they exist has gained substantial attention [23, 24, 15, 26, 2, 1, 28, 11].

The definition of a frame originates from work by Duffin and Schaeffer on non-harmonic Fourier series [10]. In general, frames can be thought of as redundant sets that generalize orthonormal bases. A frame $\{f_i\}_{i=1}^m$ for a finite d-dimensional space can be used to represent any element f in the underlying space as

$$f = \sum_{i=1}^{m} \langle f, f_i \rangle g_i = \sum_{i=1}^{m} \langle f, g_i \rangle f_i.$$

The set $\{g_i\}_{i=1}^m$ is called a *dual frame* of $\{f_i\}_{i=1}^m$. If $\{f_i\}_{i=1}^m$ is an ETF, then it has a dual that is also equiangular and tight. Roughly speaking, equiangular means that the vectors are equally spaced, and being tight means that any given f can be represented in a form that is similar to an orthogonal expansion. One can say that for ETFs, both the frame and its dual have a nice geometric structure. Unfortunately, for many pairs (m,d), ETFs either do not exist or it is unknown whether or not they exist [24]. Here we study equiangular frames that are not necessarily tight and investigate whether they can have an equiangular dual. If an equiangular dual cannot be found, the desire is to be able to classify conditions under which an equiangular frame has a dual such that the number of distinct angles among the dual frame vectors is *small*.

Some definitions and known results that will be used are collected next. We will be concerned with d-dimensional Hilbert spaces of the form $\mathbb{F}^d$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. In $\mathbb{F}^d$, a *frame* is the same as a spanning set. Given a set $\Phi = \{f_1, \ldots, f_m\}$ in $\mathbb{F}^d$, let T be a matrix whose columns are the vectors $f_1, \ldots, f_m$. T will be called the *synthesis operator* of Φ. If Φ is a frame then the $d \times d$ matrix $S = TT^*$ is called the *frame operator* of Φ. The set Φ is said to be a *tight frame* if S is a constant multiple of the identity. The set $\{\widetilde{f_i} = S^{-1} f_i\}$ is a dual frame for Φ, and is called the *canonical dual*. The frame operator of the canonical dual is S^{-1}. If for all $1 \leq i \leq m$ there is a constant c such that $\|f_i\| = c$, then Φ is called an *equal norm frame* or a *unit norm frame* if $c = 1$. For basics on frame theory, the reader is referred to [4]. A frame of m vectors in a d-dimensional space will be referred to as an (m,d) *frame*. Unless otherwise stated, it will be assumed that $m > d$.

Definition 1 (Equiangular tight frame [24, 23])

An *equiangular tight frame* (ETF) is a set $\{f_i\}_{i=1}^m$ in $\mathbb{F}^d$ satisfying

(i) $TT^* = \frac{m}{d}I$, i.e., the set is a tight frame.
(ii) $\|f_i\| = 1$, for $i = 1,\ldots,m$, i.e., the set is unit norm.
(iii) $|\langle f_i, f_j\rangle| = \sqrt{\frac{m-d}{d(m-1)}}$, $1 \le i \ne j \le m$.

The quantity $\sqrt{\frac{m-d}{d(m-1)}}$ appearing in (iii) in Definition 1 will be referred to here as the *Welch bound* . Relaxing the condition of being tight in Definition 1 gives an equiangular frame as is defined next.

Definition 2 (Equiangular frame [28, 23])

An *equiangular frame* (EF) is a set $\{f_i\}_{i=1}^m$ that spans $\mathbb{F}^d$, and satisfies

(i) $\|f_i\| = c$, for $i = 1,\ldots,m$, and some $c > 0$, i.e., the set is equal norm.
(ii) $|\langle f_i, f_j\rangle| = \alpha$, $1 \le i \ne j \le m$.

Note that in Definition 2, α equals the Welch bound only when the set is also unit norm and tight [23].

The matrix T^*T is the *Gram matrix* G of the set Φ. The (i,j)th entry of G is the inner product $\langle f_j, f_i\rangle$. If Φ is an equiangular frame (EF) then for all $i \ne j$

$$\alpha := |\langle f_i, f_j\rangle|.$$

If m vectors in $\mathbb{F}^d$ form an EF, then its Gram matrix G can be written as

$$G = cI + \alpha Q \tag{3}$$

where Q is an $m \times m$ Hermitian matrix with zero diagonal and unimodular entries elsewhere, called the *signature matrix*. For a real EF, the off-diagonal entries of Q are ± 1. By $[\lambda]^n$ it will be meant that the eigenvalue λ has multiplicity n. The Gram matrix of a frame of m vectors in $\mathbb{F}^d$ will have eigenvalues that can be written as

$$[0]^{m-d} < \lambda_1 \le \cdots \le \lambda_d,$$

and this implies that the eigenvalues of Q are [17, 15]

$$[-c/\alpha]^{m-d}, (\lambda_1 - c)/\alpha, \ldots, (\lambda_d - c)/\alpha.$$

Due to (3), the study of EFs reduces to the study of properties of Gram matrices or the corresponding signature matrices. Note that G and the corresponding frame operator S have the same nonzero eigenvalues. Thus the Gram matrix G of a tight frame has only one distinct nonzero eigenvalue.

Lemma 1 *A signature matrix Q always has at least one negative eigenvalue.*

Proof This follows from the fact that the trace of Q is zero, and the sum of the eigenvalues of a matrix is equal to the trace. □

Definition 3 ([31])

Two signature matrices Q_1 and Q_2, are *equivalent* if there exists a signed permutation matrix P such that $Q_2 = PQ_1P^{\mathrm{T}}$. Two signature matrices Q_1 and Q_2 are called *cospectral* if they have the same set of eigenvalues.

Equivalent matrices are cospectral but starting from $m = 8$ there exist examples of cospectral matrices of size $m \times m$ that are not equivalent [31].

For a given set $\{f_i\}_{i=1}^m$ in $\mathbb{F}^d$, the *angle set* is defined to be the set

$$\left\{ \left| \left\langle \frac{f_i}{\|f_i\|}, \frac{f_j}{\|f_j\|} \right\rangle \right|, 1 \leq i \neq j \leq m \right\}.$$

The number of distinct values in the angle set will be referred to as the number of angles in the frame. For an equiangular frame, the angle set has only one distinct value α, for some $\alpha \in \mathbb{R}$. Generalizing this gives the following.

Definition 4 (*k*-angle frame)

A frame $\{f_i\}_{i=1}^m$ for $\mathbb{F}^d$ is called a *k-angle frame* if the angle set has k distinct values $\{\alpha_1, \ldots, \alpha_k\}$.

The focus here is mainly on real equiangular frames, i.e, we take $\mathbb{F} = \mathbb{R}$. Nevertheless, where possible, the results have been presented in the setting of $\mathbb{F}^d$.

2 Construction and existence of equiangular frames

It is well known that there always exists a $(d+1,d)$ ETF that can be obtained from the vertices of a regular simplex [24]. Other constructions can be found in [17] for $(d+1,d)$ real ETFs, and, more recently, in [8] for both real and complex $(d+1,d)$ ETFs. However, as already mentioned in Section 1, an ETF does not exist for many pairs (m,d), and one might wish to relax the conditions of an ETF and instead look at equiangular frames that are not necessarily tight. By relaxing the requirement of being tight, one can expect EFs to be more common than ETFs. For an arbitrary size m, the existence of an EF is restricted by *Gerzon's bound*, i.e., the maximum number of equiangular lines is bounded above by $d(d+1)/2$ in $\mathbb{R}^d$, and d^2 in $\mathbb{C}^d$ [9].

As already discussed in Section 1, for any equiangular frame $\{f_i\}_{i=1}^m$ in $\mathbb{F}^d$ there is a corresponding signature matrix. See (3) and the discussion that follows. Conversely, for a given signature matrix, an EF can be constructed as discussed below. For convenience, we consider EFs that are unit norm. The method can be adapted to the construction of EFs with any given norm.

Construction of unit norm equiangular frames from signature matrices.

To construct an EF of size m, start with an $m \times m$ signature matrix Q. By Lemma 1, Q has at least one negative eigenvalue. Let the minimum eigenvalue of Q be $-\mu$, $\mu > 0$. Then

$$G = I + \frac{1}{\mu} Q$$

is the Gram matrix of a unit norm EF of m vectors in $\mathbb{R}^d$ where $m-d$ is the multiplicity of $-\mu$. Due to (1), we have

$$\frac{1}{\mu} \geq \sqrt{\frac{m-d}{d(m-1)}}$$

with equality attained if and only if the frame is also tight. From G, the actual frame vectors can be constructed via a diagonalization of G. Since G is Hermitian, it can be diagonalized by means of a unitary matrix. This means that there exists an $m \times m$ unitary matrix V and a diagonal matrix $D = \mathrm{diag}(0, \ldots, 0, \lambda_1, \ldots, \lambda_d)$, such that

$$G = VDV^*.$$

Take the last d columns of V : $\{v_{m-d+1}, \ldots, v_m\}$, and consider the $m \times d$ matrix F whose ith column is $\sqrt{\lambda_i} v_{m-d+i}$.

$$F = \begin{bmatrix} | & | & & | \\ \sqrt{\lambda_1} v_{m-d+1} & \sqrt{\lambda_2} v_{m-d+2} & \cdots & \sqrt{\lambda_d} v_m \\ | & | & & | \end{bmatrix}. \tag{4}$$

Then the rows of F are the frame vectors of a unit norm (m,d) EF whose synthesis operator is F^*. This leads to the following.

Proposition 1 *Given $m, d \in \mathbb{N}$, with $m > d > 1$. An equiangular frame of m vectors in $\mathbb{F}^d$ exists if and only if there is an $m \times m$ signature matrix Q whose minimum eigenvalue $-\mu$, $\mu > 0$, has multiplicity $m-d$. Further, if the frame is unit norm then the α in Definition 2 is $\alpha = \frac{1}{\mu} \geq \sqrt{\frac{m-d}{d(m-1)}}$ with equality holding if and only if the frame is also tight.*

For a given m, there can be $M = 2^{m(m-1)/2}$ different real $m \times m$ signature matrices, some of which might be equivalent. From these M signature matrices one can get EFs of m unit vectors in $\mathbb{R}^d$ for $2 \leq d \leq m-1$, according to Proposition 1. As already mentioned, in $\mathbb{R}^d$, $m \leq \frac{d(d+1)}{2}$ [9]. The possibilities for $3 \leq m \leq 7$, found using MATLAB, are shown in Table 1. Beyond $m = 7$, the total number of $m \times m$ signature matrices becomes too large to store and process at once.

The method of construction discussed above can be used to construct EFs whose frame operator is diagonal. This is shown below in Proposition 2. See also Examples 3.1 and 3.2. It is worthwhile to note that the result in Proposition 2 is not specific to EFs.

Table 1 Table showing number of $m \times m$ signature matrices resulting in equiangular frames in $\mathbb{R}^d$.

m	d	Total no. of $m \times m$ signature matrices	No. of signature matrices giving an (m, d) real EF
3	2	8	4
4	3	64	56
5	3	1024	192
5	4	1024	816
6	3	32768	384
6	4	32768	480
6	5	32768	31872
7	5	2097152	106528
7	6	2097152	1990560

Proposition 2 *Let G be an $m \times m$ Hermitian matrix of rank d. Then there exists a frame of m vectors in $\mathbb{F}^d$ whose Gram matrix is G such that the frame operator is diagonal.*

Proof Following a diagonalization of G, one can use the exact same notation as above, and (4), to get the frame operator to be

$$S = F^*F = \begin{bmatrix} - \sqrt{\lambda}_1 v_{m-d+1} - \\ - \sqrt{\lambda}_2 v_{m-d+2} - \\ \vdots \\ - \quad \sqrt{\lambda}_d v_m \quad - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \sqrt{\lambda}_1 v_{m-d+1} & \sqrt{\lambda}_2 v_{m-d+2} & \cdots & \sqrt{\lambda}_d v_m \\ | & | & & | \end{bmatrix}.$$

Since V is a unitary matrix

$$\langle v_i, v_j \rangle = \delta_{ij},$$

and this gives

$$S = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{bmatrix}$$

which is diagonal. □

Two frames $\Phi = \{f_j\}_{j \in J}$ and $\Psi = \{g_j\}_{j \in J}$ are *unitarily equivalent* if there is a unitary transformation U such that for all $j \in J$, $g_j = Uf_j$ [28].

Corollary 1 *Let $\Phi = \{f_i\}_{i=1}^m$ be a frame of $\mathbb{F}^d$. Then there exists a frame $\Psi = \{g_i\}_{i=1}^m$ of $\mathbb{F}^d$ such that the frame operator of Ψ is diagonal, and Φ is unitarily equivalent to Ψ.*

Proof Let G be the Gram matrix of Φ. By Proposition 2, one can construct a frame $\Psi = \{g_i\}_{i=1}^m$ such that the Gram matrix of Ψ is G, and the frame operator of Ψ is diagonal. As shown in [27], two frames are unitarily equivalent if and only if their Gram matrices are equal. Thus, Φ and Ψ are unitarily equivalent. □

3 Nontight equiangular frames and their duals

From the definition of a tight frame it is obvious that an ETF will have its canonical dual to be both equiangular and tight. The canonical dual vectors are just scaled versions of the frame vectors, and the angle set of the canonical dual will also have only one distinct value. This means that both the frame and its canonical dual have a nice structure. Phrased in terms of Definition 4, an ETF is an 1-angle frame such that the canonical dual is also an 1-angle frame. However, it is well-known that for many pairs (m,d), ETFs do not exist [24]. Here, in Section 3.2, we study some properties of the canonical dual for a nontight equiangular frame. The angle set of the canonical dual is investigated in Section 3.3.

We first give some examples of nontight equiangular frames and their canonical duals.

3.1 Examples of nontight equiangular frames

For a given d, we first investigate the number of equiangular frames of size $d+1$ that one can have in $\mathbb{R}^d$ aside from the $(d+1,d)$ ETF that is already known to exist. For a $(d+1,d)$ ETF, by the Welch bound,

$$|\langle f_i, f_j\rangle| = 1/d, \quad i \neq j.$$

Thus for a $(d+1,d)$ nontight EF, since the Welch bound is not attained,

$$|\langle f_i, f_j\rangle| > 1/d, \quad i \neq j.$$

Example 3.1 [$(d+1,d)$ equiangular frames]

1. All $(3,2)$ real unit norm equiangular frames are ETFs. This is determined from the minimum eigenvalue of each of the $2^3 = 8$ possible signature matrices of size 3×3, by checking when the multiplicity of this eigenvalue is $3-2=1$. In each of the 4 feasible cases, see Table 1, the minimum eigenvalue equals -2. The corresponding unit norm EF has common angle $\alpha = \frac{1}{2}$, which matches with the Welch bound. Thus each EF is an ETF.
2. For $d \geq 3$, there can be $(d+1,d)$ *nontight* equiangular frames. Take $d=3$. There are 64 possible real signature matrices Q of size 4×4. Out of these,

56 give equiangular frames in $\mathbb{R}^3$. The resulting unit norm equiangular frames have the common angle α to be either $\alpha = \frac{1}{3}$ or $\alpha = \frac{1}{\sqrt{5}}$. These values come from the minimum eigenvalue of the corresponding Q. The value $\alpha = \frac{1}{3}$ is the corresponding Welch bound, and corresponds to a $(4,3)$ ETF. The value $\alpha = \frac{1}{\sqrt{5}}$ comes from a $(4,3)$ nontight EF. One signature matrix Q giving rise to such a frame, and the corresponding Gram matrix G are

$$Q = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{bmatrix}.$$

As shown in Section 2, a diagonalization of G will yield a frame corresponding to this G coming from the rows of the matrix F. In this case, the rounded values of F are

$$\begin{bmatrix} 0.5257 & 0 & 0.8506 \\ -0.5257 & 0 & 0.8506 \\ 0 & -0.8506 & -0.5257 \\ 0 & 0.8506 & -0.5257 \end{bmatrix}.$$

Here F^* is the synthesis operator. The rounded values of the frame operator $S = F^*F$ are

$$\begin{bmatrix} 0.5528 & 0 & 0 \\ 0 & 1.4472 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which confirms that the frame is not tight. Note that the frame operator is diagonal, see Proposition 2. The canonical dual comes from the columns of $S^{-1}F^*$ which is the synthesis operator of the canonical dual. The rounded values of $S^{-1}F^*$ are

$$\begin{bmatrix} 0.9510 & -0.9510 & 0 & 0 \\ 0 & 0 & -0.5878 & 0.5878 \\ 0.4253 & 0.4253 & -0.2629 & -0.2629 \end{bmatrix}.$$

The matrix $[\langle \frac{\widetilde{f}_i}{\|\widetilde{f}_i\|}, \frac{\widetilde{f}_j}{\|\widetilde{f}_j\|} \rangle]$ is

$$\begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{2}{3} & 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & 1 & -\frac{2}{3} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{2}{3} & 1 \end{bmatrix}.$$

By inspection, the angle set of the canonical dual is $\{\frac{1}{6}, \frac{2}{3}\}$, and so the canonical dual of this frame has 2 angles.

Table 2 Table showing number of $(d+1)\times(d+1)$ signature matrices resulting in equiangular frames in $\mathbb{R}^d$.

d	No. of $(d+1)\times(d+1)$ signature matrices giving an EF in $\mathbb{R}^d$	No. of different possibilities of the common angle α
2	4	1
3	56	2
4	816	5
5	31872	11
6	1990560	40

For $2 \le d \le 6$, Table 2 gives the number of feasible signature matrices of size $(d+1)\times(d+1)$ that can result in equiangular frames in $\mathbb{R}^d$. The table also gives how many different values of the common angle α are possible for each d. One of these values will correspond to the Welch bound of the corresponding $(d+1,d)$ ETF. Others will correspond to the common angles of nontight equiangular frames of $d+1$ vectors in $\mathbb{R}^d$.

Example 3.2 [(n,d) equiangular frames and their duals; $n > d+1$]

There does not exist a $(5,3)$ real ETF [23]. The Welch bound for $m=5$ and $d=3$ is $\frac{1}{\sqrt{6}}$ which cannot be attained by any set of 5 vectors in $\mathbb{R}^3$. However, there exists a $(5,3)$ real EF with signature matrix

$$Q = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

The eigenvalues of Q are $-\sqrt{5}$, $-\sqrt{5}$, 0, $\sqrt{5}$, $\sqrt{5}$. The Gram matrix is

$$G = I + \frac{1}{\sqrt{5}}Q = \begin{bmatrix} 1 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{bmatrix}.$$

G has eigenvalues $\{0,0,1,2,2\}$. Thus G has two distinct nonzero eigenvalues, and the corresponding frame is nontight. For this $(5,3)$ equiangular frame, the absolute value of the inner product between any two distinct vectors is $\frac{1}{\sqrt{5}}$ as can be seen from G, and this minimizes the maximum cross correlation among all 5 vectors in $\mathbb{R}^3$ [25, 23]. To get the frame vectors of a $(5,3)$ equiangular frame, consider the SVD of G:

$$G = PDP'$$

where

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

and the rounded values of P are

$$\begin{bmatrix} 0.1995 & -0.6002 & 0.4472 & -0.3820 & -0.5041 \\ -0.5091 & -0.3752 & 0.4472 & 0.0127 & 0.6323 \\ -0.5142 & 0.3683 & 0.4472 & 0.3614 & -0.5190 \\ 0.1913 & 0.6028 & 0.4472 & -0.5974 & 0.2075 \\ 0.6324 & 0.0043 & 0.4472 & 0.6053 & 0.1833 \end{bmatrix}.$$

Following Section 2, the frame vectors of a frame corresponding to the above G are the rows of the matrix F whose rounded values are

$$\begin{bmatrix} 0.4472 & -0.5402 & -0.7129 \\ 0.4472 & 0.0180 & 0.8942 \\ 0.4472 & 0.5111 & -0.7340 \\ 0.4472 & -0.8449 & 0.2935 \\ 0.4472 & 0.8560 & 0.2592 \end{bmatrix}.$$

It can be checked that the frame operator is

$$S = F^*F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and that $FF^* = G$. Note that the frame operator is diagonal, see Proposition 2. The inverse of the frame operator is

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

The canonical dual can be obtained from the columns of $S^{-1}F^*$ whose rounded values are

$$\begin{bmatrix} 0.4472 & 0.4472 & 0.4472 & 0.4472 & 0.4472 \\ -0.2701 & 0.0092 & 0.2555 & -0.4225 & 0.4280 \\ -0.3564 & 0.4471 & -0.3670 & 0.1467 & 0.1296 \end{bmatrix}.$$

Upon calculating the Gram matrix of the canonical dual it can be seen that the canonical dual of this frame has two angles [28].

3.2 Frame properties of the canonical dual of equiangular frames

In what follows (see Corollary 2), we give a necessary and sufficient condition when the canonical dual of certain equiangular frames will also be equiangular.

Proposition 3 *[7]*

Given $\{\alpha_i\}_{i=1}^m \subset \mathbb{F}$, the following are equivalent:

(a) *There exist dual frames $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^m$ for $\mathbb{F}^d$ such that $\alpha_i = \langle f_i, g_i\rangle$ for all $1 \le i \le m$;*
(b) *$d = \sum_{i=1}^m \alpha_i$.*

Theorem 1 *[6]Let $\{f_i\}_{i=1}^m$ be an equal norm frame in $\mathbb{F}^d$. Let $\{\tilde{f}_i\}_{i=1}^m$ denote the canonical dual of $\{f_i\}_{i=1}^m$. Then $\{f_i\}_{i=1}^m$ is a tight frame if and only if $\|f_i\|\,\|\tilde{f}_i\| = \frac{d}{m}$ for $i = 1,\dots,m$.*

Proof Suppose that $\{f_i\}_{i=1}^m$ is a tight frame in $\mathbb{F}^d$ with frame bound A. Note that $\tilde{f}_i = \frac{1}{A} f_i$ for $i = 1,\dots,m$. Since $\{f_i\}_{i=1}^m$ is equal norm, Proposition 3 implies that for $i = 1,\dots,m$,

$$d = \sum_{j=1}^m \langle f_j, \tilde{f}_j\rangle = \frac{m}{A}\|f_i\|^2.$$

Thus we have

$$\|f_i\| = \sqrt{\frac{Ad}{m}},\ \|\tilde{f}_i\| = \sqrt{\frac{d}{Am}}.$$

This implies that $\|f_i\|\,\|\tilde{f}_i\| = \frac{d}{m}$ for $i = 1,\dots,m$.

Conversely, suppose that $\|f_i\|\,\|\tilde{f}_i\| = \frac{d}{m}$ for $i = 1,\dots,m$. Using Proposition 3 again, we have

$$d = \sum_{i=1}^m \langle f_i, \tilde{f}_i\rangle \le \sum_{i=1}^m \left|\langle f_i, \tilde{f}_i\rangle\right| \le \sum_{i=1}^m \|f_i\|\,\|\tilde{f}_i\| = d.$$

This implies that $|\langle f_i, \tilde{f}_i\rangle| = \|f_i\|\,\|\tilde{f}_i\|$ for $i = 1,\dots,m$. Then for $i = 1,\dots,m$, there exists a constant λ_i such that $\tilde{f}_i = \lambda_i f_i$. Since $\{f_i\}_{i=1}^m$ is equal norm, and $\|f_i\|\,\|\tilde{f}_i\| = \frac{d}{m}$ for $i = 1,\dots,m$, $|\lambda_i|$ is a constant for $i = 1,\dots,m$. Note that

$$0 \le \sum_{j=1}^m |\langle f_i, \tilde{f}_j\rangle|^2 = \langle f_i, \tilde{f}_i\rangle = \overline{\lambda}_i \|f_i\|^2$$

for $i = 1,\dots,m$. Then λ_i is a positive constant for $i = 1,\dots,m$. This implies that $\{f_i\}_{i=1}^m$ is a tight frame. □

Corollary 2 *Let $\{f_i\}_{i=1}^m$ be an equiangular frame in $\mathbb{F}^d$. Then the canonical dual $\{\tilde{f}_i\}_{i=1}^m$ is equiangular with $\|f_i\|\,\|\tilde{f}_i\| = \frac{d}{m}$ for $i = 1,\dots,m$ if and only if $\{f_i\}_{i=1}^m$ is a tight frame.*

Remark 1 Even if the canonical dual is not equiangular, due to Proposition 2, one can construct EFs such that the frame operator of the canonical dual is diagonal.

As mentioned in Section 1, equivalent signature matrices are cospectral. Thus it is obvious from (3), and the discussion that follows, that equivalent signature matrices give rise to EFs having the same common angle α, where $-\frac{1}{\alpha}$ is the minimum eigenvalue of the equivalent signature matrices. Coming to canonical duals of EFs, it is shown below in Proposition 4 that the canonical duals of certain equiangular frames corresponding to equivalent signature matrices have the same angle set. The following lemma is needed.

Lemma 2 *Let Q_1 and Q_2 be two $m \times m$ equivalent signature matrices. There exist equiangular frames $\{f_i\}_{i=1}^m$ and $\{\psi_i\}_{i=1}^m$ corresponding to Q_1 and Q_2, respectively, such that their respective frame operators S_f and S_ψ are the same.*

Proof Obtain equiangular frames $\{f_i\}_{i=1}^m$ and $\{\psi_i\}_{i=1}^m$ corresponding to Q_1 and Q_2, respectively, by following the construction of Section 2. By Proposition 2, these frames will have diagonal frame operators S_f and S_ψ, respectively. Since Q_1 and Q_2 are cospectral, the corresponding Gram matrices have the same eigenvalues. Recall that frame operators have the same nonzero eigenvalues as the corresponding Gram matrices, and so S_f and S_ψ are the same. □

Let T and F be the synthesis operators of two frames. These frames are said to have *equivalent* Gram matrices if there exist a unitary matrix U and a signed (when $\mathbb{F} = \mathbb{R}$) or phased (when $\mathbb{F} = \mathbb{C}$) permutation matrix P such that $F = UTP$. Note that frames with equivalent Gram matrices have the same angle set.

Proposition 4 *Let Q_1 and Q_2 be two $m \times m$ equivalent signature matrices. Let $\{f_i\}_{i=1}^m$ and $\{\psi_i\}_{i=1}^m$ denote equiangular frames corresponding to Q_1 and Q_2, respectively, that also satisfy Proposition 2. If the canonical dual $\{\widetilde{f_i}\}_{i=1}^m$ has angle set equal to $A = \{\alpha_1, \ldots, \alpha_k\}$, then the canonical dual $\{\widetilde{\psi_i}\}_{i=1}^m$ also has the same angle set A. Moreover, if $\{\widetilde{f_i}\}_{i=1}^m$ is equal norm then $\{\widetilde{\psi_i}\}_{i=1}^m$ is also equal norm.*

Proof By the proof of Lemma 2, $\{f_i\}_{i=1}^m$ and $\{\psi_i\}_{i=1}^m$ have the same frame operator S. Let F and T denote the synthesis operators of $\{f_i\}_{i=1}^m$ and $\{\psi_i\}_{i=1}^m$, respectively. Since the frames have equivalent signature matrices Q_1 and Q_2, they have equivalent Gram matrices, and there exists a unitary matrix U and a signed or phased permutation matrix P such that $F = UTP$. Then

$$\begin{aligned} S &= FF^* = (UTP)(UTP)^* = (UTP)P^*T^*U^* = UTT^*U^* \\ &= USU^*, \end{aligned}$$

and thus S must commute with U. This further implies that $\widetilde{U} := S^{-1}US$ is unitary. The synthesis operators of the canonical duals $\{\widetilde{f_i}\}_{i=1}^m$ and $\{\widetilde{\psi_i}\}_{i=1}^m$ are $S^{-1}F$ and $S^{-1}T$, respectively.

$$\widetilde{U}(S^{-1}T)P = S^{-1}US(S^{-1}T)P = S^{-1}UTP = S^{-1}F.$$

Thus the canonical duals yield equivalent Gram matrices, and the result follows. □

3.3 Angle sets of canonical duals of equiangular frames

In this section we investigate conditions under which an equiangular frame that is not tight can have a dual that is a k-angle frame, where it is desired that k is a small positive integer. Gram matrices of ETFs have one nonzero eigenvalue. This means that signature matrices corresponding to ETFs have two distinct eigenvalues. Examples of such signature matrices are somewhat rare [24, 31], as are ETFs. It is known that various large sets of equiangular lines have corresponding signature matrices with three distinct eigenvalues [13]. This has motivated extensive study of signature matrices with exactly three eigenvalues in [13, 31]. The results in this section and some other related results can be found in [5]. In Theorem 2 and Theorem 3 below, we analyze signature matrices with three distinct eigenvalues and study the number of possible angles in the canonical dual of any corresponding equiangular frame. It is worth noting that a result similar to Theorem 2 using strongly regular graphs is given in [28]. The following Lemma 3 will be used.

Lemma 3 *If $-\lambda_1$, λ_2, λ_3 are the three distinct eigenvalues of a signature matrix Q, ordered such that $-\lambda_1 < \lambda_2 < \lambda_3$, $\lambda_1 > 0$, then*

$$-\lambda_1 \neq \frac{\lambda_2 + \lambda_3}{2}.$$

Proof Since Q has zero trace, it must have at least one positive eigenvalue, so $\lambda_3 > 0$. Now, if $-\lambda_1 = \frac{\lambda_2+\lambda_3}{2}$, then $\lambda_2 + \lambda_3 < 0$; thus $-\lambda_1 < \lambda_2 < \lambda_2 + \lambda_3 < \frac{\lambda_2+\lambda_3}{2} < 0$, which is a contradiction. $\square$

The following lemma can be proved by a direct calculation of the characteristic polynomial of Q.

Lemma 4 *If Q is an $m \times m$ signature matrix whose off-diagonal entries are all 1 then Q has two distinct eigenvalues: $m-1$ with multiplicity 1, and -1 with multiplicity $m-1$.*

An eigenvector is said to be *regular* if its entries are ± 1. In what follows, $\mathbf{1}$ denotes the vector whose each entry is 1, and J is the matrix whose entries are all 1. The (i, j)th entry of a matrix A will be denoted by $A(i, j)$.

Theorem 2 *Let Q be an $m \times m$ signature matrix with three distinct eigenvalues $-\lambda_1, \lambda_2, \lambda_3$, ordered such that $-\lambda_1 < \lambda_2 < \lambda_3$, with $\lambda_1 > 0$. Let Φ denote any corresponding equiangular frame of m vectors in $\mathbb{R}^d$. Then the following hold.*
(a) Suppose that λ_2 or λ_3 is a simple eigenvalue with a regular eigenvector, and the multiplicity of $-\lambda_1$ is r. If the sum of this simple eigenvalue and λ_1 is not m, then the canonical dual is an equal norm 2-angle nontight frame, and $d = m - r$.
(b) Suppose that the minimum eigenvalue $-\lambda_1$ is simple with a regular eigenvector. Then the canonical dual is an equal norm 2-angle nontight frame. In this case, $d = m - 1$.

Proof The Gram matrix of a tight frame can have only one nonzero eigenvalue. Since Q has three distinct eigenvalues, G must have two distinct nonzero eigenvalues and so Φ is a not a tight frame. Thus the dual is also not tight in both (a) and (b).

Let P_1, P_2, and P_3 denote the orthogonal projections onto the eigenspaces of $-\lambda_1$, λ_2, and λ_3, respectively. By the Spectral Theorem $Q = -\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$, where $P_1 + P_2 + P_3 = I$, and for $i \neq j$, $P_i P_j = 0$. The Gram matrix of Φ is

$$G = I + \frac{1}{\lambda_1} Q = \frac{\lambda_1 + \lambda_2}{\lambda_1} P_2 + \frac{\lambda_1 + \lambda_3}{\lambda_1} P_3. \tag{5}$$

The Gram matrix of the canonical dual is the pseudo inverse of G [3], and given by

$$G^\dagger = \frac{\lambda_1}{\lambda_1 + \lambda_2} P_2 + \frac{\lambda_1}{\lambda_1 + \lambda_3} P_3. \tag{6}$$

(a) Since the multiplicity of the minimum eigenvalue of Q is $r = m - d$, the value of d is obvious. Without loss of generality assume that λ_3 is simple with a regular eigenvector v. Then $P_3 = \frac{1}{\|v\|^2} v v^{\mathrm{T}} = \frac{1}{m} v v^{\mathrm{T}}$. Note that the diagonal entries of P_3 are all equal to $\frac{1}{m}$. This implies, from (5), that P_2 also has constant diagonal. Therefore, in (6), $G^\dagger$ must have constant diagonal too, implying that the canonical dual is equal norm.

Equating the off-diagonal entries of G in (5) gives

$$\pm \frac{1}{\lambda_1} = \frac{\lambda_1 + \lambda_2}{\lambda_1} P_2(i,j) + \frac{\lambda_1 + \lambda_3}{\lambda_1} P_3(i,j),$$

or,

$$P_2(i,j) = \left[\pm 1 - (\lambda_1 + \lambda_3) P_3(i,j)\right] \frac{1}{\lambda_1 + \lambda_2}. \tag{7}$$

Using (6), (7), and the fact that the off-diagonal entries of P_3 are $\pm \frac{1}{m}$

$$G^\dagger(i,j) = \pm \frac{\lambda_1}{(\lambda_1 + \lambda_2)^2} + \left(\pm \frac{1}{m}\right) \left[\frac{\lambda_1}{\lambda_1 + \lambda_3} - \frac{\lambda_1(\lambda_1 + \lambda_3)}{(\lambda_1 + \lambda_2)^2} \right], \quad i \neq j. \tag{8}$$

From (8), the absolute values of the off-diagonal entries of $G^\dagger$ can take only two values, and this can be justified as follows. The expression inside the bracket in the second term on the right side of (8) cannot be zero due to Lemma 3. Since Q has three distinct eigenvalues, Q cannot have all its off-diagonal entries equal to 1 or all equal to -1 due to Lemma 4. Thus the off-diagonal entries of G take both values $\pm \frac{1}{\lambda_1}$. If P_3 and G have the exact same or the exact opposite sign distribution in their off-diagonal entries, then, from (7) and the given assumption, the off-diagonal entries of P_2 will equal $\pm c$ for some constant c and have the same sign distribution as that of P_3 or G. In that case, $P_2 P_3 \neq 0$, which contradicts the Spectral Theorem. Thus P_3 and G cannot have the exact same or the exact opposite sign distribution. All this suggests that the *absolute* values of the off-diagonal entries of $G^\dagger$ take only

two values, and the canonical dual is a 2-angle frame.
(b) Suppose that $-\lambda_1$ is a simple eigenvalue. Since the multiplicity of the minimum eigenvalue is 1, the frame is in $\mathbb{R}^{m-1}$.

If $\mathbf{1}$ is an eigenvector for $-\lambda_1$ then $Q = -\lambda_1 \frac{J}{m} + \lambda_2 P_2 + \lambda_3 P_3$. Using the fact that $\frac{J}{m} + P_2 + P_3 = I$ in (5) gives

$$G = \frac{\lambda_1+\lambda_2}{\lambda_1} I - \frac{\lambda_1+\lambda_2}{\lambda_1}\frac{J}{m} + \frac{\lambda_3-\lambda_2}{\lambda_1} P_3.$$

The Gram matrix of the dual then becomes

$$G^{\dagger} = \frac{\lambda_1}{\lambda_1+\lambda_2} P_2 + \frac{\lambda_1}{\lambda_1+\lambda_3} P_3 = \frac{\lambda_1}{\lambda_1+\lambda_2}(I - \frac{J}{m}) + \lambda_1 (\frac{1}{\lambda_1+\lambda_3} - \frac{1}{\lambda_1+\lambda_2}) P_3.$$

Equating the diagonal and off-diagonal entries of G and $G^{\dagger}$, one can conclude that in this case the dual is 2-angle and equal norm.

Next suppose that $-\lambda_1$ is a simple eigenvalue with a regular eigenvector v that is not $\mathbf{1}$. This time

$$G = \frac{\lambda_1+\lambda_2}{\lambda_1} I - \frac{\lambda_1+\lambda_2}{\lambda_1} P_1 + \frac{\lambda_3-\lambda_2}{\lambda_1} P_3. \tag{9}$$

and

$$G^{\dagger} = \frac{\lambda_1}{\lambda_1+\lambda_2} I - \frac{\lambda_1}{\lambda_1+\lambda_2} P_1 + \frac{\lambda_1(\lambda_2-\lambda_3)}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)} P_3. \tag{10}$$

Note that $P_1 = \frac{1}{m} v v^{\mathrm{T}}$ with diagonal entries all equal to $\frac{1}{m}$. Thus, in (9), the matrices G, I, and P_1 all have constant diagonal. This implies that P_3 also has constant diagonal. Using this in (10) shows that $G^{\dagger}$ also has constant diagonal, i.e., the canonical dual frame is equal norm.

Solving for P_3 in (9), and using the fact that the off-diagonal entries of P_1 are $\pm\frac{1}{m}$, gives

$$P_3(i,j) = \frac{1}{\lambda_3-\lambda_2}\left[\pm 1 \pm \frac{\lambda_1+\lambda_2}{m}\right], \quad \text{for } i \neq j. \tag{11}$$

Substituting (11) in (10), gives for $i \neq j$

$$G^{\dagger}(i,j) = \mp\frac{1}{m}\lambda_1 \frac{2\lambda_1+\lambda_2+\lambda_3}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)} \mp \frac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)}. \tag{12}$$

Due to Lemma 3, the first term on the right of (12) cannot be zero. Thus

$$G^{\dagger}(i,j) = \mp\frac{1}{m} C_2 \mp C_3,$$

where C_2 and C_3 are nonzero constants. This implies that the canonical dual is a 2-angle frame. $\square$

Suppose that a signature matrix Q with three distinct eigenvalues has an irrational eigenvalue $\lambda + \sqrt{\mu}$. Then the other two eigenvalues of Q are $\lambda - \sqrt{\mu}$ and some $k \in \mathbb{Z}$ [13, 31]. The following result proved in [12] will be used.

Lemma 5 (Corollary 5.6 [12])

Let Q be an $m\times m$ signature matrix with three distinct eigenvalues, at least one of which is irrational. If m is odd then the eigenvalues of Q are

$$[-\sqrt{m}]^{(m-1)/2},\ [0]^1,\ [\sqrt{m}]^{(m-1)/2}.$$

Theorem 3 *Let Q be an $m\times m$ signature matrix with three distinct eigenvalues, at least one of which is irrational. Let Φ denote any corresponding equiangular frame of m vectors in $\mathbb{R}^d$.*
(i) If m is odd, then the canonical dual is an equal norm frame with at most $m-1$ angles, and $d=\frac{m+1}{2}$.
(ii) Let Q have eigenvalues

$$[-k]^{m-2n}, [a-\sqrt{b}]^n, [a+\sqrt{b}]^n$$

with minimum eigenvalue $-k$, $k\in\mathbb{Z}^+$, $a\in\mathbb{Q}$, $b\in\mathbb{Q}^+$, and $m-2n>1$. Let the number of distinct moduli in the irrational part of the projection matrix of the eigenspace of either $a+\sqrt{b}$ or $a-\sqrt{b}$ be p. Then the canonical dual has at most $2p$ angles, and $d=2n$.

Proof (i) Due to Lemma 5, the eigenvalues of Q in this case are

$$[-\sqrt{m}]^{(m-1)/2},\ [0]^1,\ [\sqrt{m}]^{(m-1)/2}.$$

Since the multiplicity of the minimum eigenvalue is $\frac{m-1}{2}$, the value of d is given by

$$d=m-\frac{m-1}{2}=\frac{m+1}{2}.$$

Denote the projection matrices of $-\sqrt{m}$ and $\sqrt{m}$ by P_1 and $P_{\hat{1}}$, respectively. By the Spectral Theorem, $Q=-\sqrt{m}P_1+\sqrt{m}P_{\hat{1}}$. Note that due to properties of eigenvectors corresponding to irrational eigenvalues, P_1 and $P_{\hat{1}}$ are irrational conjugates of each other, and can be written as $P_1=P_a+P_b,\ P_{\hat{1}}=P_a-P_b$ where the (i,j)th entries of P_a and P_b are given by

$$\begin{aligned}&P_a(i,j)=a_{ij}\in\mathbb{Q}\\ &P_b(i,j)=0\quad\text{or}\quad \pm\sqrt{b_{ij}},\ b_{ij}\ \in\mathbb{Q},\ b_{ij}\text{ not a perfect square.}\end{aligned}$$

It follows that $Q=-2\sqrt{m}P_b$. The Gram matrix of Φ and the Gram matrix of the canonical dual are given by

$$G=I+\frac{1}{\sqrt{m}}Q=I-P_1+P_{\hat{1}}=I-2P_b, \tag{13}$$

$$G^{\dagger}=I-P_1-\frac{1}{2}P_{\hat{1}}=I-\frac{3}{2}P_a-\frac{1}{2}P_b, \tag{14}$$

respectively. Using the relations $P_1^2 = P_1$, $P_{\hat{1}}^2 = P_{\hat{1}}$, $P_{\hat{1}}P_1 = 0$, $P_1 P_{\hat{1}} = 0$, one gets $P_a = 2P_b^2$ and thus

$$G^{\dagger} = I - 3P_b^2 - \frac{1}{2}P_b. \tag{15}$$

The off-diagonal entries of G are $\pm\frac{1}{\sqrt{m}}$, and its diagonal entries are all equal to 1. Equating the (i,j)th entries of the matrices in (13) then gives

$$P_b(i,j) = \begin{cases} 0 & \text{if } i = j \\ \pm\frac{1}{2\sqrt{m}} & \text{if } i \neq j. \end{cases}$$

Let $\beta := \frac{1}{2\sqrt{m}}$. The diagonal entries of P_b^2 are then all equal to $(m-1)\beta^2$. Since the diagonal entries of P_b are all equal to zero, this means from (15) that $G^{\dagger}$ has constant diagonal, and that the canonical dual is equal norm.
The absolute values of the off-diagonal entries of P_b^2 can take *at most* $\frac{m-1}{2}$ distinct values given by

$$\{(m-2)\beta^2, (m-4)\beta^2, \ldots, \beta^2\}.$$

This combined with the fact that $I - \frac{1}{2}P_b$ can take the values $1 \pm \frac{\beta}{2}$ means that the absolute values of the off diagonal entries of $G^{\dagger}$ can take at most $m-1$ distinct values.

(ii) Now the multiplicity of the minimum eigenvalue is $m-2n$, and so d equals $2n$. Let P_2 and $P_{\hat{2}}$ denote the projection matrices of $a+\sqrt{b}$ and $a-\sqrt{b}$, respectively. As in part(i), these can be written as $P_2 = P_a + P_b$, $P_{\hat{2}} = P_a - P_b$, Let P_1 denote the projection matrix of $-k$. Since $m-2n > 1$, $\mathbf{1}$ is not a basis for the eigenspace of $-k$. Thus $P_1 \neq \frac{1}{m}J$, and the number of angles in the canonical dual cannot be determined by Theorem 2. By the Spectral Theorem

$$Q = -kP_1 + (a+\sqrt{b})(P_a + P_b) + (a-\sqrt{b})(P_a - P_b).$$

Using $P_1 + P_2 + P_{\hat{2}} = I$ gives

$$G = I + \frac{1}{k}Q = \frac{2}{k}\left((k+a)P_a + \sqrt{b}P_b\right).$$

The Gram matrix of the canonical dual is the pseudo inverse

$$G^{\dagger} = \frac{2k}{(k+a)^2 - b}\left[\frac{kG}{2} - 2\sqrt{b}P_b\right].$$

The result then follows from the fact that since Φ is equiangular, the off-diagonal entries of G are either $\frac{1}{k}$ or $-\frac{1}{k}$. □

Then existence of signature matrices satisfying the conditions of Theorem 2 and Theorem 3 has been discussed in [12] and [31].

Due to the algebraic properties of signature matrices [31], one cannot expect to generalize the above results to any arbitrary number of distinct eigenvalues of Q. In the context of regular graphs, signature matrices with four eigenvalues are discussed in [32], and for this case, Theorem 3 can be extended as follows.

Theorem 4 *Let Q be an $m \times m$ signature matrix with four distinct eigenvalues*

$$[a-\sqrt{b}]^n, [a+\sqrt{b}]^n, [-k_1]^{m-2n-1}, [k_2]^1$$

with minimum eigenvalue $-k_1$, where k_1, $k_2 \in \mathbb{Z}^+$, $a \in \mathbb{Q}$, $b \in \mathbb{Q}^+$. Suppose that $\mathbf{1}$ *is an eigenvector of Q corresponding to k_2. Let the number of distinct moduli in the purely irrational part of the projection matrix of the eigenspace of either $a+\sqrt{b}$ or $a-\sqrt{b}$ be p. If Φ is any equiangular frame corresponding to Q then the canonical dual is a frame in $\mathbb{R}^{2n+1}$ having at most $2p$ angles.*

Proof The projection matrix of the eigenspace of k_2 is $\frac{J}{m}$. The spectral decomposition of Q is

$$Q = -k_1 P_1 + k_2 \frac{J}{m} + (a+\sqrt{b})P_2 + (a-\sqrt{b})P_{\hat{2}}.$$

The proof then follows in an identical manner as Theorem 3 by noting that the off-diagonal entries of J are all one. □

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