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Welch Bounds for Cross Correlation of Subspaces and Generalizations

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Lower bounds on the maximal cross correlation between vectors in a set were first given by Welch and then studied by several others. In this work, this is extended to obtaining lower bounds on the maximal cross correlation between subspaces of a given Hilbert space. Two different notions of cross correlation among spaces have been considered. The study of such bounds is done in terms of fusion frames, including generalized fusion frames. In addition, results on the expectation of the cross correlation among random vectors have been obtained.

Keywords: Frames; fusion frames; generalized frames; random frames; subspaces; tight frames; Welch bounds

AMS Subject Classification: 15Axx; 42C15

1. Introduction

1.1. *Background and motivation*

Given a set of M unit vectors $\{f_i\}_{i=1}^M$ in $\mathbb{C}^N$, Welch [23] gave a family of lower bounds on the maximal cross correlation among the vectors. Given an integer $K \geq 1$, Welch showed that

$$\max_{i \neq j} |\langle f_i, f_j \rangle|^{2K} \geq \frac{1}{M-1}\left[\frac{M}{\binom{N+K-1}{K}} - 1\right]. \tag{1}$$

In fact, these bounds were obtained as a consequence of the following inequality:

$$\sum_{i=1}^{M}\sum_{j=1}^{M} |\langle f_i, f_j \rangle|^{2K} \geq \frac{M^2}{\binom{N+K-1}{K}}. \tag{2}$$

These bounds have become a standard tool in waveform design for both communications and radar. Motivated by applications in communications (e.g., CDMA), several authors have studied conditions under which the Welch bounds in (1) and (2) are attained. Sets satisfying the lower bounds for $K = 1$ also arise in other application contexts as in quantum information processing and coding theory [12, 14, 16–18].

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The original derivation of the bounds as done by Welch was analytical. Nonetheless, these bounds have a geometric character. The geometric perspective has been used in [7, 20, 22], and the characterization of sets of vectors satisfying the lower bounds has been done in terms of *frames*. It seems natural to extend the geometric study of this family of bounds to the maximal cross correlation among *subspaces* of a given Hilbert space. Another possible generalization is to consider vectors $\{f_i\}$ indexed by a continuous set, or *random* vectors drawn from some probability distribution. In the latter case, the *expectation* of the cross correlation can be derived and the deviation of the cross correlation from the expectation can be studied. These extensions of the original Welch bounds in (1) and (2) are discussed in this work. The main objective here is to put on common ground lower bounds on the maximal cross correlation among vectors indexed by a discrete or continuous set, subspaces, and random vectors in a (discrete) set with the underlying space being some finite dimensional Hilbert space. The author has recently become aware of work on p-fusion frames [1] where such lower bounds also arise. Even though the fusion frame potential is used here to derive the required lower bounds, p-fusion frames are not studied in this work.

1.2. *Notation and preliminaries*

Let $\mathcal{H}$ be a Hilbert space and let $X = \{f_k\}_{k\in\mathcal{K}}$ be a collection of vectors in $\mathcal{H}$. Then X is said to be a *frame* for $\mathcal{H}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that for any $h \in \mathcal{H}$,

$$A\|h\|^2 \leq \sum_{k\in\mathcal{K}} |\langle h, f_k\rangle|^2 \leq B\|h\|^2. \tag{3}$$

The constants A and B are called the *frame bounds*. If $A = B$, the frame is said to be *tight*. Tight frames for which $A = B = 1$ are called *Parseval* frames.

The map $F : \mathcal{H} \to \ell_2(\mathcal{K})$ given by $F(h) = \{\langle h, f_k\rangle\}_{k\in\mathcal{K}}$ is called the *analysis operator*. The *synthesis operator* is the adjoint map $F^* : \ell_2(\mathcal{K}) \to \mathcal{H}$, given by

$$F^*(\{a_k\}) = \sum_{k\in\mathcal{K}} a_k f_k.$$

The *frame operator* $S : \mathcal{H} \to \mathcal{H}$ is given by $S = F^*F$. This means that for all $h \in \mathcal{H}$,

$$Sh = \sum_{k\in\mathcal{K}} \langle h, f_k\rangle f_k. \tag{4}$$

Every h has an expansion formula in terms of the frame vectors given by

$$h = \sum_{k\in\mathcal{K}} \langle h, S^{-1}f_k\rangle f_k = \sum_{k\in\mathcal{K}} \langle h, f_k\rangle S^{-1}f_k.$$

Here $\{S^{-1}f_k\}_{k\in\mathcal{K}}$ is also a frame and is called the *dual frame*. Due to (4), one can write the frame condition (3) as

$$A\langle h, h\rangle \leq \langle Sh, h\rangle \leq B\langle h, h\rangle. \tag{5}$$

For a tight frame, the frame operator is just a constant multiple of the identity, i.e., $S = A\mathcal{I}$, where $\mathcal{I}$ is the identity map. For the general theory on frames one can refer to [5], [8].

The above concepts on frames can be generalized in several ways. If the family $\{f_t\}_{t\in\mathcal{K}}$ is indexed by a continuum rather than a discrete set then what results is called a *generalized frame* [13] or *continuous frame* [5]. Let $(\mathcal{K}, \mathcal{B}, \mu)$ be a measure space that plays the role of the index set. Let $f : \mathcal{K} \to \mathcal{H}$ be a μ-measureable function where $t \in \mathcal{K}$ is mapped to f_t in $\mathcal{H}$. The set $\{f_t\}_{t\in\mathcal{K}}$ is a *generalized frame* for $\mathcal{H}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that for all $h \in \mathcal{H}$,

$$A\|h\|^2 \leq \int_{\mathcal{K}} |\langle h, f_t\rangle|^2 \; d\mu(t) \leq B\|h\|^2. \tag{6}$$

Let $\Pi_v : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto the one dimensional subspace spanned by $v \in \mathcal{H}$, i.e., $\Pi_v(h) = \langle h, v\rangle v$. The frame operator of $\{f_t\}_{t\in\mathcal{K}}$ is then

$$S_\mu(h) = \int_{\mathcal{K}} \Pi_{f_t}(h) \; d\mu(t).$$

The summands in (4) are all rank one projections. Therefore, another way to generalize the concept of a frame is to consider sums of projections whose ranks are greater than or equal to one. Let $\{W_k\}_{k\in\mathcal{K}}$ be a sequence of subspaces in $\mathcal{H}$ and P_k be the orthogonal projection of $\mathcal{H}$ onto W_k. Then

$$\left\langle \sum_{k\in\mathcal{K}} P_k h, h \right\rangle = \sum_{k\in\mathcal{K}} \|P_k h\|^2, \quad \forall h \in \mathcal{H}.$$

Considering $\sum_{k\in\mathcal{K}} P_k h$, instead of Sh in (5), a fusion frame can be defined as follows [4]. A family of closed subspaces $\{W_k\}_{k\in\mathcal{K}}$ is called a *fusion frame* for $\mathcal{H}$, with respect to weights $\{w_k\}_{k\in\mathcal{K}}$, $w_k > 0$, if there exist constants $0 < A \leq B < \infty$ such that for all $h \in \mathcal{H}$,

$$A\|h\|^2 \leq \sum_{k\in\mathcal{K}} w_k^2 \|P_k h\|^2 \leq B\|h\|^2. \tag{7}$$

A combination of generalized frames and fusion frames leads naturally to the definition of a *generalized fusion frame* [15]. Let $(\mathcal{K}, \mathcal{B}, \mu)$ be a measure space. Let $w : \mathcal{K} \to \mathbb{R}^+$ and $f : \mathcal{K} \to \mathcal{P}(\mathcal{H})$, where $\mathcal{P}(\mathcal{H})$ is the space of orthogonal projections on $\mathcal{H}$. Let w_t and f_t be the images of t under the functions w and f, respectively. The collection $\{w_t, f_t\}_{t\in\mathcal{K}}$ is a generalized fusion frame if there exist constants $0 < A \leq B < \infty$ such that for all $h \in \mathcal{H}$,

$$A\|h\|^2 \leq \left\langle \int_{\mathcal{K}} w_t^2 f_t(h) d\mu(t), h \right\rangle \leq B\|h\|^2.$$

A great overview of generalizations of frames can be found in [15].

The frame potential [2] of a set $\{f_i\}_{i=1}^M$ of vectors is defined to be the sum

$$\sum_{i,j=1}^{M} |\langle f_i, f_j\rangle|^2.$$

It can be shown [2] that the minimizers of the frame potential are tight frames, meaning that unit-normed tight frames attain the lower bound in (2) when $K = 1$. Note that the frame potential of $\{f_i\}_{i=1}^M$ is the trace of the square of the frame operator S. Considering $\sum_{k\in\mathcal{K}} P_k$ instead of the frame operator S, the fusion frame potential of $\{W_k\}_{k\in\mathcal{K}}$ is defined as [3]

$$\mathrm{FFP}(\{W_k\}_{k\in\mathcal{K}}) := \mathrm{tr}\left(\sum_{k\in\mathcal{K}} P_k\right)^2 = \sum_{i\in\mathcal{K}}\sum_{j\in\mathcal{K}} \mathrm{tr}(P_iP_j). \tag{8}$$

The K-fold tensor product $V^{\otimes K}$ of an N-dimensional vector space V is a vector space spanned by elements of the form $v_1 \otimes \cdots \otimes v_K$ where each $v_i \in V$ [9, 19]. The vector $v_1\otimes\cdots\otimes v_K$ is called a tensor and has N^K coordinates $(v_1^{(\ell_1)}, v_2^{(\ell_2)}, \ldots, v_K^{(\ell_K)})$, where $\ell_i = 1, 2, \ldots, N,\ i = 1, 2, \ldots, K$, and $v_i^{(\ell)}$ denotes the ℓ^{th} coordinate of the vector v_i. A choice of basis $\{e_1, \ldots, e_N\}$ for V gives rise to a basis for $V^{\otimes K}$ consisting of the N^K product elements $e_{i_1\ldots i_K} \equiv e_{i_1} \otimes\cdots\otimes e_{i_K}$, $1 \le i_1, \ldots, i_K \le N$. In particular, $V^{\otimes K}$ has dimension N^K.

The space of symmetric K-tensors associated with V, denoted $\mathrm{Sym}^K(V)$, is the subspace of $V^{\otimes K}$ spanned by those tensors which remain fixed under permutation. Specifically, denote by S_K the symmetric group on K symbols and define an action of S_K on $V^{\otimes K}$ by

$$A_\sigma(v_1 \otimes \cdots \otimes v_K) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(K)}.$$

Then $\mathrm{Sym}^K(V)$ consists of all elements of $V^{\otimes K}$ such that $A_\sigma(v_1 \otimes \cdots \otimes v_K) = v_1 \otimes \cdots \otimes v_K$ for all $\sigma \in S_K$ (see Chapter 10 of [19]). If V has dimension n then

$$\dim\ \mathrm{Sym}^K(V) = \binom{n+k-1}{k}.$$

$\mathrm{Sym}^K(V)$ has a natural inner product with the property

$$\left\langle v^{\otimes K}, w^{\otimes K}\right\rangle_{\mathrm{Sym}^K(V)} = \langle v, w\rangle_V^K. \tag{9}$$

1.3. *Outline*

In Section 2, a lower bound on the cross correlation of subspaces of a given Hilbert space is obtained by using the inner product of orthogonal projections onto the subspaces. In Section 2.1 an alternate way of calculating such lower bounds is discussed by using the principal angles between subspaces. Conditions under which the lower bounds will be attained are discussed in Section 2.2. Welch bounds for generalized fusion frames and random frames are discussed in sections 3 and 4, respectively.

2. Welch bounds for subspaces

Let W_1 and W_2 be two subspaces of an N-dimensional Hilbert space $\mathcal{H}$. Let P_1 and P_2 be the orthogonal projections onto W_1 and W_2, respectively. Define the inner product

$$\langle P_1, P_2\rangle := \mathrm{tr}(P_1^* P_2) = \mathrm{tr}(P_1 P_2).$$

This inner product of P_1 and P_2 can be used as a measure of the cross correlation between the corresponding subspaces W_1 and W_2. In the following, a finite number of M subspaces $\{W_\ell\}_{\ell=1}^M$ of the Hilbert space $\mathcal{H}$ is considered. The orthogonal projection of $\mathcal{H}$ onto W_j is denoted by P_j. Let $\dim(W_j) = L_j$. Then $\dim(\mathcal{R}(P_j)) = \dim(W_j) = L_j$. If P is an orthogonal projection then $\mathrm{tr}(P) = \dim(\mathcal{R}(P))$. Thus $\mathrm{tr}(P_j) = L_j$. The fusion frame potential defined in (8) then becomes

$$\mathrm{FFP}(\{W_\ell\}_{\ell=1}^M) = \sum_{i,j=1}^M \mathrm{tr}\left(P_i P_j\right) = \sum_{i,j=1}^M \langle P_i, P_j\rangle.$$

Proposition 2.1 *[3] Let P_ℓ be the orthogonal projection of an N-dimensional Hilbert space $\mathcal{H}$ onto a subspace W_ℓ, $1 \le \ell \le M$. If L_ℓ is the trace of P_ℓ, then*

$$FFP(\{W_\ell\}_{\ell=1}^M) = \sum_{i,j=1}^M \langle P_i, P_j\rangle \ge \frac{1}{N}\left(\sum_{\ell=1}^M L_\ell\right)^2.$$

Theorem 2.2 *[First Welch bound for subspaces] Under the assumptions of Proposition 2.1,*

$$\max_{\ell\neq\ell'}\langle P_\ell, P_{\ell'}\rangle \ge \frac{\sum_{\ell=1}^M L_\ell}{M(M-1)}\left[\frac{1}{N}\sum_{\ell=1}^M L_\ell - 1\right].$$

Proof. Using Proposition 2.1 and the fact that the maximum of a set of numbers is greater than or equal to the average, the following holds.

$$\max_{\ell\neq\ell'}\langle P_\ell, P_{\ell'}\rangle \ge \frac{1}{M(M-1)}\sum_{\ell\neq\ell'}\langle P_\ell, P_{\ell'}\rangle = \frac{1}{M(M-1)}\sum_{\ell\neq\ell'}\mathrm{tr}(P_{\ell'}P_\ell)$$
$$= \frac{1}{M(M-1)}\left[\sum_{\ell,\ell'=1}^M \mathrm{tr}(P_{\ell'}P_\ell) - \sum_{\ell=1}^M \mathrm{tr}(P_\ell^2)\right] \ge \frac{1}{M(M-1)}\left[\frac{1}{N}\left(\sum_{\ell=1}^M L_\ell\right)^2 - \sum_{\ell=1}^M \mathrm{tr}(P_\ell)\right]$$
$$= \frac{1}{M(M-1)}\left[\frac{1}{N}\left(\sum_{\ell=1}^M L_\ell\right)^2 - \sum_{\ell=1}^M L_\ell\right] = \frac{\sum_{\ell=1}^M L_\ell}{M(M-1)}\left[\frac{1}{N}\sum_{\ell=1}^M L_\ell - 1\right].$$

□

Theorem 2.3 (Bounds for higher order fusion frame potential) *Under the assumptions of Proposition 2.1, let $P_\ell^{(K)}$ denote the orthogonal projection of $Sym^K(\mathcal{H})$ onto $Sym^K(W_\ell)$, for some integer $K > 1$. Then*

$$FFP\{Sym^K(W_\ell)\}_{\ell=1}^M := tr\left(\sum_{\ell=1}^M P_\ell^{(K)}\right)^2 \ge \frac{\left(\sum_{\ell=1}^M \binom{L_\ell+K-1}{K}\right)^2}{\binom{N+K-1}{K}}. \qquad (10)$$

Proof. Under the assumptions, $\mathcal{R}(\sum_{\ell=1}^{M} P_\ell^{(K)}) \subseteq \mathrm{Sym}^K(\mathcal{H})$. It is known that $\dim(\mathrm{Sym}^K(\mathcal{H})) = \binom{N+K-1}{K}$ and therefore

$$\dim\left(\mathcal{R}\left(\sum_{\ell=1}^{M} P_\ell^{(K)}\right)\right) \leq \dim\left(\mathrm{Sym}^K(\mathcal{H})\right) = \binom{N+K-1}{K}.$$

Let $\sum_{\ell=1}^{M} P_\ell^{(K)}$ have D nonzero eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_D$. Then $D \leq \binom{N+K-1}{K}$. The fusion frame potential of $\{\mathrm{Sym}^K(W_\ell)\}_{\ell=1}^{M}$ is

$$\mathrm{FFP}\{\mathrm{Sym}^K(W_\ell)\}_{\ell=1}^{M} := \mathrm{tr}\left(\sum_{\ell=1}^{M} P_\ell^{(K)}\right)^2 = \sum_{n=1}^{D} \lambda_n^2 \geq \frac{\left(\sum_{n=1}^{D} \lambda_n\right)^2}{D} \geq \frac{\left(\sum_{n=1}^{D} \lambda_n\right)^2}{\binom{N+K-1}{K}}.$$

Further,

$$\sum_{n=1}^{D} \lambda_n = \mathrm{tr}\left(\sum_{\ell=1}^{M} P_\ell^{(K)}\right) = \sum_{\ell=1}^{M} \mathrm{tr}\left(P_\ell^{(K)}\right) = \sum_{\ell=1}^{M} \dim\left(\mathcal{R}\left(P_\ell^{(K)}\right)\right)$$
$$= \sum_{\ell=1}^{M} \dim\left(\mathrm{Sym}^K(W_\ell)\right) = \sum_{\ell=1}^{M} \binom{L_\ell + K - 1}{K}$$

and therefore,

$$\mathrm{FFP}\{\mathrm{Sym}^K(W_\ell)\}_{\ell=1}^{M} \geq \frac{\left(\sum_{\ell=1}^{M} \binom{L_\ell+K-1}{K}\right)^2}{\binom{N+K-1}{K}}.$$

□

Let $\{f_i\}_{i=1}^{M_1}$ and $\{g_i\}_{i=1}^{M_2}$ be frames for the subspaces W_1 and W_2, respectively. The respective dual frames are written as $\{\widetilde{f}_i\}_{i=1}^{M_1}$ and $\{\widetilde{g}_i\}_{i=1}^{M_2}$. If P_1 and P_2 are orthogonal projections of $\mathcal{H}$ onto W_1 and W_2, respectively, then for any f in $\mathcal{H}$,

$$P_1(f) = \sum_{i=1}^{M_1} \langle f, \widetilde{f}_i \rangle f_i, \;\; P_2(f) = \sum_{i=1}^{M_2} \langle f, \widetilde{g}_i \rangle g_i.$$

Using the fact that $\langle P_1, P_2 \rangle = \mathrm{tr}(P_1^* P_2) = \mathrm{tr}(P_1 P_2)$, the following lemma follows by direct calculation.

Lemma 2.4 *Let W_1 and W_2 be two subspaces of $\mathcal{H}$. Let $\{f_i\}_{i=1}^{M_1}$ and $\{g_i\}_{i=1}^{M_2}$ be frames for W_1 and W_2, respectively. If P_1 and P_2 are orthogonal projections of $\mathcal{H}$ onto W_1 and W_2, respectively, then*

$$tr(P_1 P_2) = \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \langle g_j, f_i \rangle \langle \widetilde{f}_i, \widetilde{g}_j \rangle. \tag{11}$$

In analogy with the classical Welch bounds on $\max_{i \neq j} |\langle f_i, f_j \rangle|^{2K}$ for a set of vectors $\{f_i\}$, the lower bounds for $\max_{\ell \neq \ell'} \langle P_\ell, P_{\ell'} \rangle^K$ for some integer $K > 1$ is calculated next.

THEOREM 2.5 (Higher order Welch bounds for subspaces) *Let $K > 1$ be an integer. Let $\mathcal{H}$ be an N-dimensional Hilbert space and $\{W_\ell\}_{\ell=1}^M$ be M subspaces of $\mathcal{H}$. Assume that for each subspace W_ℓ, there exists a Parseval frame $X_\ell = \{f_i\}$ such that the set $X_\ell^{(K)} = \{f_i^{\otimes K}\}$ is a tight frame for $Sym^K(W_\ell)$, and the product of the frame bounds of any pair $X_i^{(K)}$ and $X_j^{(K)}$ is greater than or equal to 1. Then, under the notation of Theorem 2.3,*

(a)

$$\sum_{\ell,\ell'=1}^{M} \langle P_\ell, P_{\ell'}\rangle^K \geq \sum_{\ell,\ell'=1}^{M} tr(P_\ell^{(K)} P_{\ell'}^{(K)}) \geq \frac{\left(\sum_{\ell=1}^{M}\binom{L_\ell+K-1}{K}\right)^2}{\binom{N+K-1}{K}},$$

(b)

$$\max_{\ell\neq\ell'} \langle P_\ell, P_{\ell'}\rangle^K \geq \frac{1}{M(M-1)}\left[\frac{\left(\sum_{\ell=1}^{M}\binom{L_\ell+K-1}{K}\right)^2}{\binom{N+K-1}{K}} - \sum_{\ell=1}^{M}(L_\ell)^K\right]. \tag{12}$$

Proof. Consider the case $K = 2$. By using (11), and the same notation as in Lemma 2.4,

$$\begin{aligned}
\langle P_1, P_2\rangle^2 = (\mathrm{tr}(P_1P_2))^2 &= \left(\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\langle g_j, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_j\rangle\right)^2 \\
&= \sum_{i=1}^{M_1}\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_\ell\rangle\right)^2 + \sum_{i\neq j}\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_\ell\rangle\right)\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_j\rangle\langle \widetilde{f}_j, \widetilde{g}_\ell\rangle\right) \\
&= \sum_{i=1}^{M_1}\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_i\rangle^2\langle \widetilde{f}_i, \widetilde{g}_\ell\rangle^2 + \sum_{r\neq s}\langle g_r, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_r\rangle\langle g_s, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_s\rangle\right) + \\
&\qquad \sum_{i\neq j}\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_i\rangle\langle \widetilde{f}_i, \widetilde{g}_\ell\rangle\right)\left(\sum_{\ell=1}^{M_2}\langle g_\ell, f_j\rangle\langle \widetilde{f}_j, \widetilde{g}_\ell\rangle\right) \\
&= \sum_{i=1}^{M_1}\sum_{\ell=1}^{M_2}\langle g_\ell^{\otimes 2}, f_i^{\otimes 2}\rangle\langle \widetilde{f}_i^{\otimes 2}, \widetilde{g}_\ell^{\otimes 2}\rangle + S_1 + S_2.
\end{aligned}$$

where S_1 and S_2 will be computed explicitly. Let $\{f_i\}_{i=1}^{M_1}$ and $\{g_i\}_{i=1}^{M_2}$ be Parseval frames for W_1 and W_2, respectively, such that $\{f_i^{\otimes 2}\}_{i=1}^{M_1}$ and $\{g_i^{\otimes 2}\}_{i=1}^{M_2}$ are tight frames with bounds A_1 and A_2, with $A_1A_2 \geq 1$. Then, $\widetilde{f}_i^{\otimes 2} = f_i^{\otimes 2} = A_1\widetilde{f_i^{\otimes 2}}$ and $\widetilde{g}_i^{\otimes 2} = g_i^{\otimes 2} = A_2\widetilde{g_i^{\otimes 2}}$. If $P_1^{(2)}$ and $P_2^{(2)}$ denote the orthogonal projections onto $\mathrm{Sym}^2(W_1)$ and $\mathrm{Sym}^2(W_2)$, respectively, then Lemma 2.4 can be used for $\mathrm{Sym}^2(W_1)$

and $\mathrm{Sym}^2(W_2)$ to obtain

$$\begin{aligned}\langle P_1, P_2\rangle^2 &= A_1A_2\sum_{i=1}^{M_1}\sum_{\ell=1}^{M_2}\langle g_\ell^{\otimes 2}, f_i^{\otimes 2}\rangle\langle \widetilde{f_i^{\otimes 2}}, \widetilde{g_\ell^{\otimes 2}}\rangle + S_1 + S_2\\ &= A_1A_2\mathrm{tr}(P_1^{(2)}P_2^{(2)}) + S_1 + S_2\\ &\geq \mathrm{tr}(P_1^{(2)}P_2^{(2)}) + S_1 + S_2 \end{aligned} \tag{13}$$

where

$$S_1 = \sum_i\sum_{r\neq s}\langle g_r, f_i\rangle\langle\widetilde{f}_i, \widetilde{g}_r\rangle\langle g_s, f_i\rangle\langle\widetilde{f}_i, \widetilde{g}_s\rangle = \sum_i\sum_{r\neq s}|\langle g_r, f_i\rangle|^2|\langle g_s, f_i\rangle|^2$$

and

$$S_2 = \sum_{i\neq j}\left(\sum_{r=1}^{M_2}\langle g_r, f_i\rangle\langle\widetilde{f}_i, \widetilde{g}_r\rangle\right)\left(\sum_{s=1}^{M_2}\langle g_s, f_j\rangle\langle\widetilde{f}_j, \widetilde{g}_s\rangle\right) = \sum_{i\neq j}\sum_{r=1}^{M_2}|\langle f_i, g_r\rangle|^2\sum_{s=1}^{M_2}|\langle f_j, g_s\rangle|^2.$$

Note that $S_1 \geq 0$, $S_2 \geq 0$, and are equal to zero when W_1 and W_2 belong to the orthogonal complement of each other. Thus

$$\langle P_1, P_2\rangle^2 \geq \mathrm{tr}(P_1^{(2)}P_2^{(2)}).$$

In general, for $K > 1$, let $P_1^{(K)}$ and $P_2^{(K)}$ denote the orthogonal projections of $\mathcal{H}$ onto $\mathrm{Sym}^K(W_1)$ and $\mathrm{Sym}^K(W_2)$, respectively. Starting with Parseval frames $\{f_i\}_{i=1}^{M_1}$ and $\{g_i\}_{i=1}^{M_2}$ of W_1 and W_2, assume that $\{f_i^{\otimes K}\}_{i=1}^{M_1}$ and $\{g_i^{\otimes K}\}_{i=1}^{M_2}$ are tight frames with frame bounds A_1 and A_2, respectively, satisfying $A_1A_2 \geq 1$. Then

$$\begin{aligned}\langle P_1, P_2\rangle^K = [\mathrm{tr}(P_1P_2)]^K &= \left(\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\langle g_j, f_i\rangle\langle\widetilde{f}_i, \widetilde{g}_j\rangle\right)^K\\ &= \sum_{i=1}^{M_1}\left(\sum_{j=1}^{M_2}\langle g_j, f_i\rangle\langle\widetilde{f}_i, \widetilde{g}_j\rangle\right)^K + T_1\\ &= \sum_{i=1}^{M_1}\left(\sum_{j=1}^{M_2}\langle g_j, f_i\rangle^K\langle\widetilde{f}_i, \widetilde{g}_j\rangle^K + T_2\right) + T_1\\ &= \sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\langle g_j^{\otimes K}, f_i^{\otimes K}\rangle\langle\widetilde{f}_i^{\otimes K}, \widetilde{g}_j^{\otimes K}\rangle + T_3\\ &\geq \mathrm{tr}(P_1^{(K)}P_2^{(K)}). \end{aligned} \tag{14}$$

The terms T_1, T_2, and T_3 in the above consist of sums involving $|\langle g_j, f_i\rangle|^2$, $1 \leq i \leq M_1$, $1 \leq j \leq M_2$, and are therefore bounded below by zero, the lower bound of zero being attained by these sums when the subspaces W_1 and W_2 are in the orthogonal complement of each other.

From (14), the result in (a) follows due to Theorem 2.3 by noting that $\operatorname{tr}\left(\sum_{\ell=1}^{M} P_\ell^{(K)}\right)^2 = \sum_{\ell,\ell'=1}^{M} \operatorname{tr}(P_\ell^{(K)} P_{\ell'}^{(K)})$.

(b) Once again, using the fact that the maximum of a set of numbers is greater than or equal to the average, one gets

$$\begin{aligned}
\max_{\ell\neq\ell'}\langle P_\ell, P_{\ell'}\rangle^K &\geq \frac{1}{M(M-1)}\sum_{\ell,\ell'}\langle P_\ell, P_{\ell'}\rangle^K = \frac{1}{M(M-1)}\sum_{\ell,\ell'}[\operatorname{tr}P_{\ell'}P_\ell]^K \\
&= \frac{1}{M(M-1)}\left[\sum_{\ell,\ell'=1}^{M}[\operatorname{tr}(P_{\ell'}P_\ell)]^K - \sum_{\ell=1}^{M}[\operatorname{tr}(P_\ell^2)]^K\right] \\
&= \frac{1}{M(M-1)}\left[\sum_{\ell,\ell'=1}^{M}[\operatorname{tr}(P_{\ell'}P_\ell)]^K - \sum_{\ell=1}^{M}[\operatorname{tr}(P_\ell)]^K\right] \\
&= \frac{1}{M(M-1)}\left[\sum_{\ell,\ell'=1}^{M}[\operatorname{tr}(P_{\ell'}P_\ell)]^K - \sum_{\ell=1}^{M}L_\ell^K\right] \\
&\geq \frac{1}{M(M-1)}\left[\sum_{\ell,\ell'=1}^{M}\operatorname{tr}(P_\ell^{(K)}P_{\ell'}^{(k)}) - \sum_{\ell=1}^{M}L_\ell^K\right] \quad \text{(due to 14).}
\end{aligned}$$

Using either Theorem 2.3 or part (a),

$$\max_{\ell\neq\ell'}\langle P_\ell, P_{\ell'}\rangle^K \geq \frac{1}{M(M-1)}\left[\frac{\left(\sum_{\ell=1}^{M}\binom{L_\ell+K-1}{K}\right)^2}{\binom{N+K-1}{K}} - \sum_{\ell=1}^{M}L_\ell^K\right].$$

□

Remark 1 It is important to note that in general it is challenging to construct a tight frame $X = \{f_1, \ldots, f_m\}$ for a space W such that the set of pure tensors $X^{(K)} = \{f_1^{\otimes K}, \ldots, f_m^{\otimes K}\}$ is a tight frame for $\operatorname{Sym}^K(W)$. However, for the case $K = 2$, these are given by mutually unbiased bases [14].

2.1. *Bounds on cross correlation using principal angles*

In this section, an alternate notion is used to quantify the cross correlation between subspaces. This is done by using the principal angles between subspaces and the corresponding principal vectors. It is seen that in this case the results reduce to the standard Welch bounds for vectors. Nonetheless, the idea is interesting enough to be included here.

Let W_1 and W_2 be two subspaces of a Hilbert space $\mathcal{H}$. Let the dimensions of W_1 and W_2 be p and q, respectively, where it can be assumed that $p \geq q$. The *principal angles* $\theta_1, \ldots, \theta_q \in [0, \pi/2]$ between W_1 and W_2 can be defined recursively as follows [11]. Let

$$\cos(\theta_1) = \max_{u\in W_1}\max_{v\in W_2}\langle\frac{u}{\|u\|}, \frac{v}{\|v\|}\rangle =: \langle u_1, v_1\rangle. \tag{15}$$

For $k > 1$, let

$$\cos(\theta_k) = \max_{u \in W_1} \max_{v \in W_2} \langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \rangle =: \langle u_k, v_k \rangle \tag{16}$$

subject to:

$$\langle u, u_i \rangle = 0, \quad \text{and} \quad \langle v, v_i \rangle = 0, \quad i = 1, \dots, k-1$$

The vectors $\{u_i\}_{i=1}^q$ and $\{v_i\}_{i=1}^q$ are called the *principal vectors*. Note that the principal angles satisfy $0 \leq \theta_1 \leq \cdots \leq \theta_q \leq \pi/2$. The following may be used as a measure of the cross correlation among subspaces.

$$\mathrm{Corr}_{W_1 W_2} := \cos(\theta_q).$$

Let $\{W_\ell\}_{\ell=1}^M$ be a sequence of M subspaces in an N-dimensional Hilbert space $\mathcal{H}$. Noting that the cross correlation between subspaces is studied here through M principal vectors, one from each subspace, this unsurprisingly reduces to the ordinary Welch bounds on inner product between vectors:

$$\max_{i \neq j} \mathrm{Corr}_{W_i W_j}^{2K} \geq \frac{1}{M-1} \left[\frac{M}{\binom{N+K-1}{K}} - 1 \right] \tag{17}$$

for $K \geq 1$. If the dimension of $\mathcal{H}$ is greater than or equal to the number of subspaces, that is, $N \geq M$, then the bounds in (17) become trivial.

2.2. *Equality in Welch bounds for subspaces*

It seems natural to study conditions on subspaces that would result in attaining the lower bounds. For a fixed integer $K \geq 1$, equality is attained in (10) of Theorem 2.3 if and only if all the eigenvalues of $\sum_{\ell=1}^M P_\ell^{(K)}$ are equal, i.e., when $\{\mathrm{Sym}^K(W_\ell)\}_{\ell=1}^M$ is a tight fusion frame for some subspace of $\mathrm{Sym}^K(\mathcal{H})$. Besides, in order to attain the equality, it is also required that $\bigoplus_{\ell=1}^M \mathrm{Sym}^K(W_\ell) = \mathrm{Sym}^K(\mathcal{H})$. To consider equality in (12), it is necessary for each subspace W_i to have a Parseval frame $X = \{f_n\}$ such that the set $X^{(K)} = \{f_n^{\otimes K}\}$ is a tight frame for $\mathrm{Sym}^K(W_i)$ and it has already been noted that such sets X are in general challenging to construct [14]. For equality in (12), it is then required that $W_\ell^{(K)}$ and $W_{\ell'}^{(K)}$ are in the orthogonal complement of each other, for $\ell \neq \ell'$. In (17) of Section 2.1, since the notion of cross correlation considered reduces to that of the usual Welch bounds on inner product of vectors, equality is attained if and only if the principal vector from each subspace, corresponding to the largest principal angle, forms an equiangular tight frame [7].

3. Welch bounds for generalized fusion frames

With V an N-dimensional subspace of $\mathcal{H}$, denote by S^{N-1} the set of unit vectors in V. For each $x \in S^{N-1}$, there is an associated projector $\Pi_x : V \to \mathrm{span}(x)$ (i.e., onto the one-dimensional subspace spanned by x) given by

$$\Pi_x(v) = \langle x, v \rangle \, x.$$

Since $\Pi_x = \Pi_{e^{i\theta}x}$ for any $\theta \in [0, 2\pi)$, the collection of projectors Π_x is parameterized by the complex projective space $\mathbb{CP}^{N-1}$. For $K \geqslant 1$, consider $\mathrm{Sym}^K(V)$. In this setting, the projector $\Pi_{x^{\otimes K}}$ maps $\mathrm{Sym}^K(V)$ onto the one-dimensional subspace spanned by the tensor power $x^{\otimes K}$ with $x \in S^{N-1}$. Direct calculation using (9) yields

$$\Pi_{x^{\otimes K}} = \Pi_x^{\otimes K},$$

and, for $v \in V$,

$$\Pi_x^{\otimes K} v^{\otimes K} = \langle x, v\rangle^K x^{\otimes K}$$

This collection of projectors is parameterized by $\mathbb{CP}^{N-1}$. Corresponding to each $x \in \mathbb{CP}^{N-1}$, choosing a representative unit vector in V yields a collection of unit vectors

$$X^{(K)}_{\mathbb{CP}^{N-1}} = \{u_x^{\otimes K} |\ u_x \in V, x \in \mathbb{CP}^{N-1}\}$$

Given a normalized measure μ on $\mathbb{CP}^{N-1}$, $X^{(K)}_{\mathbb{CP}^{N-1}}$ becomes a generalized frame for $\mathrm{Sym}^K(V)$ with frame operator $S^{(K)}_\mu : \mathrm{Sym}^K(V) \to \mathrm{Sym}^K(V)$ by

$$S^{(K)}_\mu = \int_{\mathbb{CP}^{N-1}} \Pi_{x^{\otimes K}} \, d\mu(x).$$

Theorem 3.1 *[7] Let μ be a normalized measure on $\mathbb{CP}^{N-1}$ and let $X_{\mathbb{CP}^{N-1}}$ be a generalized frame for an N-dimensional subspace V of a Hilbert space $\mathcal{H}$. Then for all $K \geq 1$,*

$$\iint_{\mathbb{CP}^{n-1}} |\langle x, y\rangle|^{2K} d\mu(x) d\mu(y) \geqslant \frac{1}{\binom{N+K-1}{K}}, \tag{18}$$

with equality if and only if $(X^{(K)}_{\mathbb{CP}^{N-1}}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^K(V)$.

In the spirit of the work presented here, one can try to look for similar bounds for generalized fusion frames. Let $(\mathcal{K}, \mathcal{B}, \mu)$ be a measure space and $\{w_t, f_t\}_{t\in\mathcal{K}}$ a generalized fusion frame. Considering the weights w_t to be all equal to one, the *generalized fusion frame potential* of $\{f_t\}_{t\in\mathcal{K}}$ can be defined as

$$\mathrm{GFFP}(\{f_t\}_{t\in\mathcal{K}}) := \mathrm{tr}\left(\int_{\mathcal{K}} f_x \ d\mu(x)\right)^2. \tag{19}$$

Then

$$\mathrm{GFFP}(\{f_t\}_{t\in\mathcal{K}}) = \iint_{\mathcal{K}} \mathrm{tr}\,(f_x f_y) \ d\mu(x) \ d\mu(y).$$

Theorem 3.2 *Let V be an N dimensional subspace of $\mathcal{H}$ and let $\{f_t\}_{t\in\mathcal{K}}$ be a generalized fusion frame for V with respect to a measure space $(\mathcal{K}, \mathcal{B}, \mu)$. Then*

$$GFFP(\{f_t\}_{t\in\mathcal{K}}) \geq \frac{\left(\int_{\mathcal{K}} tr(f_x)\, d\mu(x)\right)^2}{N}.$$

The proof of Theorem 3.2 is a direct consequence of the Cauchy Schwarz Inequality and the properties of the trace function.

4. Welch bounds for random frames

Let $\{Y_{k\ell}\}_{k,\ell\in\mathbb{Z}}$ be a doubly indexed sequence of random variables that are independent identically distributed (i. i. d.) according to the Gaussian or normal distribution with mean zero and variance σ^2, written as $Y_{k\ell} \sim N(0,\sigma^2)$.[1] Let $X_{k\ell} = e^{\frac{2\pi i}{\epsilon}Y_{k\ell}}$, where ϵ is a fixed constant to be chosen.[2] Define a vector $f_j \in \mathbb{C}^n$ as [3]

$$f_j = \frac{1}{\sqrt{n}}\begin{bmatrix} X_{j1} \\ X_{j2} \\ \vdots \\ X_{jn} \end{bmatrix}.$$

Consider the set $\{f_1,\ldots,f_M\}$. This is a set of random vectors in $\mathbb{C}^n$. The analysis operator of this set is a random matrix with independent entries. The singular values of the analysis operator are the eigenvalues of the frame operator. By using known results on the non-asymptotic distribution of singular values of random matrices [21], the frame properties of such sets of random vectors can be studied [6].[4] To estimate the expectation of the cross correlation among such random vectors first note that $|X_{k\ell}| = 1$ for all k and ℓ. Let ϕ denote the characteristic function of a random variable. If $Y \sim N(0,\sigma^2)$, then

$$\phi(t) = e^{-\frac{\sigma^2}{2}t^2}. \tag{20}$$

Therefore, for integers $r \neq s$ and $\ell \neq \ell'$,

$$E[X_{r\ell}X_{s\ell'}\overline{X}_{r\ell'}\ \overline{X}_{s\ell}] = \left[\phi\left(\frac{2\pi}{\epsilon}\right)\right]^4 = e^{-2\sigma^2(2\pi/\epsilon)^2}. \tag{21}$$

Using (21),

$$\begin{aligned}
&E[|\langle f_r, f_s\rangle|^2] = E[\langle f_r, f_s\rangle\overline{\langle f_r, f_s\rangle}] \\
&= \frac{1}{n^2}E\left[\sum_{\ell=1}^{n} X_{r\ell}\overline{X}_{s\ell}\sum_{\ell'=1}^{n}\overline{X}_{r\ell'}X_{s\ell'}\right] \\
&= \frac{1}{n^2}E\left[\sum_{\ell=1}^{n}|X_{r\ell}|^2|X_{s\ell}|^2 + \sum_{\ell\neq\ell'} X_{r\ell}\overline{X}_{r\ell'}\ \overline{X}_{s\ell}X_{s\ell'}\right] \\
&= \frac{1}{n^2}(n + n(n-1)e^{-2\sigma^2(2\pi/\epsilon)^2}) = \frac{1}{n} + \frac{n-1}{n}e^{-2\sigma^2(2\pi/\epsilon)^2}. \qquad (22)
\end{aligned}$$

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[1] The results here are for the Gaussian random variable, but other random variables can be used.

[2] The specific construction given here comes from constructing random sequences with arbitrarily low expected autocorrelation outside the origin [6].

[3] The dimension here is taken to be n instead of N to avoid confusion with the notation for a normal random variable.

[4] Note that these random frames are different from the so called probabilistic frames in [10].

To estimate the deviation of $|\langle f_r, f_s\rangle|^2$ from its expectation one can define a Doob martingale and use Azuma's Inequality as follows. Fix r, s, $r \neq s$. Let

$$\alpha_\ell := X_{r\ell}\overline{X}_{s\ell}.$$

Let

$$U_j = E\left[|\langle f_r, f_s\rangle|^2 |\ \alpha_1, \ldots, \alpha_j\right].$$

Under this definition,

$$U_0 = E\left[|\langle f_r, f_s\rangle|^2\right], \text{and } U_n = |\langle f_r, f_s\rangle|^2.$$

Then $\{U_0, \ldots, U_n\}$ gives a Doob martingale. Further,

$$|U_j - U_{j-1}| \leq 2\frac{n-1}{n}e^{-2\sigma^2\left(\frac{2\pi}{\epsilon}\right)^2} = C.$$

By Azuma's Inequality,

$$P\left[\left||\langle f_r, f_s\rangle|^2 - E\left[|\langle f_r, f_s\rangle|^2\right]\right| \geq \lambda\right] = P\left[|U_n - U_0| \geq \lambda\right] \leq e^{-\lambda^2/2\sum_1^n C^2} = e^{-\lambda^2/2nC^2}.$$

For an integer $k > 1$, the expected value of $|\langle f_r, f_s\rangle|^{2k}$ is calculated next.

$$\begin{aligned}|\langle f_r, f_s\rangle|^{2k} &= \frac{1}{n^{2k}}\left(\sum_{\ell=1}^n |\alpha_\ell|^2 + \sum_{\ell\neq\ell'} \alpha_\ell\overline{\alpha}_{\ell'}\right)^k \\ &= \frac{1}{n^{2k}}\sum_{p=0}^k \binom{k}{p}\left(\sum_{\ell=1}^n |\alpha_\ell|^2\right)^{k-p}\left(\sum_{\ell\neq\ell'}\alpha_\ell\overline{\alpha}_{\ell'}\right)^p.\end{aligned}$$

Using the fact that $|\alpha_\ell|^2 = 1$ gives

$$E\left(|\langle f_r, f_s\rangle|^{2k}\right) = \frac{1}{n^k} + \frac{1}{n^k}\sum_{p=1}^k \binom{k}{p}\frac{1}{n^p}E\left(\sum_{\ell\neq\ell'}\alpha_\ell\overline{\alpha}_{\ell'}\right)^p. \tag{23}$$

The summation in the second term on the right side in (23) can be expanded as

$$\left(\sum_{\ell\neq\ell'}\alpha_\ell\overline{\alpha}_{\ell'}\right)^p = \sum_{\ell\neq\ell'}\alpha_\ell^p\overline{\alpha}_{\ell'}^p + \sum_{\ell\neq\ell'\neq m\neq m'}\alpha_\ell^{p-1}\overline{\alpha}_{\ell'}^{p-1}\alpha_m\overline{\alpha}_{m'} + \ldots. \tag{24}$$

By using independence and the characteristic function of a normal random variable (see (20)), the expectation of each term in the summations on the right side of (24) can be calculated. For the first term in (24),

$$E(\alpha_\ell^p\overline{\alpha}_{\ell'}^p) = E(X_{r\ell}^p\overline{X}_{s\ell}^p\overline{X}_{r\ell'}^p X_{s\ell'}^p) = (\phi(\frac{2\pi}{\epsilon}p))^4 = e^{-2\sigma^2p^2(\frac{2\pi}{\epsilon})^2}.$$

Thus this term, and similarly other terms that arise in the summation in (23), can be made arbitrarily small with the choice of ϵ. The deviation of $|\langle f_r, f_s\rangle|^{2k}$ from the expectation can be estimated by an application of Azuma's Inequality for Doob martingales as done for $k = 1$ above.

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