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Geometry of the Welch Bounds

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Abstract

A geometric perspective involving Grammian and frame operators is used to derive the entire family of Welch bounds. This perspective unifies a number of observations that have been made regarding tightness of the bounds and their connections to symmetric k-tensors, tight frames, homogeneous polynomials, and t-designs. In particular, a connection has been drawn between sampling of homogeneous polynomials and frames of symmetric k-tensors. It is also shown that tightness of the bounds requires tight frames. The lack of tight frames of symmetric k-tensors in many cases, however, leads to consideration of sets that come as close as possible to attaining the bounds. The geometric derivation is then extended in the setting of generalized or continuous frames. The Welch bounds for finite sets and countably infinite sets become special cases of this general setting.

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1. Introduction

1.1. Background

In a brief but important 1974 paper [34], L. R. Welch considered the situation of unit vectors $\{x_1, \ldots, x_m\}$ in $\mathbb{C}^n$ with $m > n$. For the maximal cross correlation $c_{\max} = \max_{i \neq j} |\langle x_i, x_j \rangle|$ among the vectors, he developed a family of lower bounds on $c_{\max}^{2k}$, parameterized by $k \geq 1$, given by

$$c_{\max}^{2k} \geqslant \frac{1}{m-1}\left[\frac{m}{\binom{n+k-1}{k}} - 1\right]. \tag{1}$$

He described the implications of these bounds in the design of sequences having desirable correlation properties for multichannel communications applications. In the years following their original derivation, the Welch bounds became a standard tool in waveform design for both communications and radar. Welch obtained inequality (1) as a corollary to a more fundamental one:

$$\sum_{i=1}^{m}\sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}}. \tag{2}$$

Indeed, most recent work on this topic recognizes (2) as Welch's main result and refers to these inequalities as the Welch bounds.

Some variations on this basic result have been noted. For example, relaxing the unit-norm assumption [32] to allow the x_i to be any non-zero vectors yields

$$\frac{\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k}}{\left(\displaystyle\sum_{i=1}^{m} ||x_i||^{2k}\right)^2} \geqslant \frac{1}{\binom{n+k-1}{k}}.$$

In any case, the bounds given in (2) are at the heart of the subject and will be the focus of attention here.

Although Welch's original derivation was analytical, several subsequent authors have noted that the Welch bounds have a geometric character. Geometric derivations of the first Welch bound, i.e., for $k = 1$, were published in 2003

by Strohmer and Heath [27] and Waldron [32]. Shapiro gave a similar argument in unpublished notes a few years earlier [25]. In this paper, the geometric perspective is extended to derive the entire family of Welch bounds.

For $k = 1$, conditions under which the Welch bound is attained in (1) or (2) have been studied explicitly by several authors [18, 23, 35, 10, 32, 27]. In this case, design methods for sets that meet the bound with equality have been proposed [18, 23, 27, 35, 10, 14]. In this context, the motivation for identifying such sets has generally involved communications (e.g., CDMA), though they also have application in waveform design for radar and sonar.

Welch bound equality sets arise in other application contexts, including quantum information processing and coding theory, as well as in purely mathematical settings. In quantum information theory, symmetric informationally complete positive operator-valued measures (SIC-POVMs) provide a general model for quantum measurement. Relationships between SIC-POVMs, complex projective t-designs, mutually unbiased bases, tight frames, and sets attaining the Welch bounds with equality has been noted in many places [17, 21, 22]. The treatment of Welch bounds for $k \geq 1$ in (2) and sets that satisfy them with equality, from the perspective of mutually unbiased bases and t-designs, is found in [17, 21, 22]. Related results on complex projective t-designs, as seen from a more mathematical perspective, is given in other places, for example [13].

This paper gives *frame* conditions for equality in the Welch bounds in both (1) and (2), for all $k \geq 1$, see Section 3, and also comments on conditions under which these bounds are relevant. However, constructing such sets for $k > 1$ is challenging. It is known, from results, including some in the literature mentioned above, that such sets do not exist in many cases. Here the idea of frame potential energy [1] is used as the basis for the notion of "Approximate Welch Bound Equality" sets.

The existence of isometries between certain spaces of homogeneous polynomials and symmetric tensors is well established in published work [26]. It is shown here that there is a connection between reconstruction of a homogeneous polynomial from its samples and tight frames of symmetric tensors.

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Strohmer and Heath [27] developed Welch-like bounds in infinite-dimensional settings, whereas this paper gives new results for infinite collections of vectors that form a frame for a finite-dimensional space. This involves formulation of the Welch bounds in the setting of generalized frames. The results generated here (Section 5) seem related to results appearing elsewhere in the literature, notably [22].

1.2. Outline

The foundation of the relationship between the Welch bounds and symmetric k-tensors is elucidated in the derivation of the Welch bounds given in Section 2. It is shown that the derivation can be done using either the Grammian or the frame operator. Section 3 contains discussion on the construction and existence of Welch Bound Equality (WBE) sets, Maximal Welch Bound Equality (MWBE) sets, and Approximate Welch Bound Equality (AWBE) sets. Section 4 addresses the connection to homogeneous polynomials and gives a sampling result for homogeneous polynomials. An extension to generalized frames, which subsumes both the finite and countably infinite frame cases, is presented in Section 5. The section concludes with some remarks relating tight generalized frames to Haar measures and linking homogeneous polynomials and t-designs.

1.3. Notation and Terminology

For $x = [x^{(1)} \; \cdots \; x^{(n)}]^\mathsf{T}$ and $y = [y^{(1)} \; \cdots \; y^{(n)}]^\mathsf{T}$ in $\mathbb{C}^n$, their inner product will be denoted by

$$\langle x, y \rangle = \sum_{j=1}^{n} \overline{x^{(j)}} y^{(j)}$$

where the bar denotes complex conjugate; i.e., the inner product is conjugate linear in its first argument and linear in its second argument. The corresponding convention will be used for inner products in other complex Hilbert spaces. Given a finite frame $\Phi = \{x_1, ..., x_m\}$ for an n-dimensional complex vector space V, the function $F : V \to \ell_2(\{1, \ldots, m\}) = \mathbb{C}^m$ given by

$$F(w) = [\langle x_1, w \rangle \; \ldots \; \langle x_m, w \rangle]^\mathsf{T}$$

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will be called the *Bessel map* associated with Φ, while $\mathcal{F} = F^*F : V \to V$ (i.e., the composition of the adjoint of F with F) will be called the *frame operator* associated with Φ. This terminology will be carried over to the setting of generalized frames in Section 5. For the fundamentals on frame theory the reader is referred to [6, 7].

The space of symmetric k-tensors ([26], [11]) associated with a vector space V is denoted by $\mathrm{Sym}^k(V)$. $\mathrm{Sym}^k(V)$ is spanned by the tensor powers $v^{\otimes k}$ where $v \in V$. If V has dimension n then

$$\dim \ \mathrm{Sym}^k(V) = \binom{n+k-1}{k}.$$

$\mathrm{Sym}^k(V)$ has a natural inner product with the property

$$\left\langle v^{\otimes k}, w^{\otimes k} \right\rangle_{\mathrm{Sym}^k(V)} = \langle v, w \rangle_V^k \ . \tag{3}$$

The identity map from the vector space V to itself is denoted by $\mathcal{I}_V$.

2. The Welch bounds

2.1. The first Welch bound

The bound in (2) with $k = 1$ has received, by far, the most attention in the literature. As noted in Section 1.1, geometric proofs of this particular bound have appeared in published work and were known as early as 1998 [9]. The "first Welch bound", i.e., for $k = 1$ in (2), is derived in this section. This derivation introduces the essential geometric foundations for obtaining the general case, which is carried out in the following section.

The following lemma will be key in deriving the Welch bounds.

Lemma 2.1. *Let W be a finite dimensional subspace of a complex Hilbert space $\mathbb{H}$ and let $T : \mathbb{H} \to \mathbb{H}$ be a positive semidefinite linear operator whose range is W.[1] Denote $n = \dim W$ and let $\lambda_1, \ldots, \lambda_n$ be the non-zero eigenvalues of T.*

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[1] Most past work on Welch bounds is set explicitly in $\mathbb{C}^n$. It is useful in what follows to take the slightly more abstract perspective set forth here.

Then the Hilbert-Schmidt (Frobenius) norm of T satisfies

$$||T||^2 \geqslant \frac{1}{n} \left(\sum_{i=1}^{n} |\lambda_i| \right)^2 = \frac{|\text{tr } T|^2}{\dim W} \ . \tag{4}$$

Equality holds if and only if all the eigenvalues are equal to each other in which case

$$T = \frac{|\text{tr } T|^2}{\dim W} \mathcal{I}_W.$$

The above lemma follows by the Cauchy-Schwarz inequality and the fact that the Frobenius norm of T satisfies

$$||T||^2 = \sum_{i=1}^{n} |\lambda_i|^2.$$

Theorem 2.2 (First Welch Bound). *Suppose that $X = \{x_1, \ldots, x_m\}$ is a set of unit vectors in $\mathbb{H}$ that span a subspace V of dimension n with $m > n$. Then*

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^2 \geqslant \frac{m^2}{n} \ . \tag{5}$$

Proof. Let F be the Bessel map on V associated with X. Then the Grammian $\mathcal{G} = FF^*$ is an operator of rank n on $\mathbb{C}^m$ whose Frobenius norm is

$$||\mathcal{G}|| = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}}$$

and whose trace is m. Further, the rank of $\mathcal{G}$ is exactly n, so it operates non-trivially on a subspace $W \subset \mathbb{C}^m$ of that dimension. Thus applying (4) of Lemma 2.1 to $\mathcal{G}$ yields the Welch bound (5).

A "dual" argument is obtained by considering the frame operator $\mathcal{F} = F^*F :$ $\mathbb{H} \to \mathbb{H}$. The non-zero eigenvalues of $\mathcal{F}$ are identical to those of $\mathcal{G}$, so its trace and rank are also equal to those of $\mathcal{G}$. So (4) of Lemma 2.1 applied to $\mathcal{F}$ also yields the result. □

2.2. Higher-order Welch bounds

Alternatives to Welch's original analytical derivation of the bounds (2) for $k > 1$ do not seem to appear in published literature. In fact, these cases also

follow from (4) by considering either k-fold Hadamard products [15] or tensor products.[2]

Theorem 2.3 (Higher Order Welch Bounds). *Suppose that $X = \{x_1, \ldots, x_m\}$ is a set of unit vectors in $\mathbb{H}$ that span a subspace V of dimension n. Then for all integers $k \geqslant 1$*

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}} \ . \tag{6}$$

Proof. (i) The left-hand side of (6) is the Hilbert-Schmidt norm of the k-fold Hadamard product [15] $\mathcal{G}^{\circ k}$ of the Grammian $\mathcal{G}$ associated with X. From work done in [19], the rank of $\mathcal{G}^{\circ k}$ is at most $\binom{n+k-1}{k}$. [3] The Schur product theorem [15, 19] ensures that $\mathcal{G}^{\circ k}$ is positive semidefinite. Since tr $\mathcal{G}^{\circ k} = \sum_{i=1}^{m} ||x_i||^{2k} = m$, (4) gives

$$||\mathcal{G}^{\circ k}||^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}} \ .$$

(ii) Alternatively, consider the space $\mathrm{Sym}^k(V)$ where V is the n-dimensional span of X. This space has dimension $\binom{n+k-1}{k}$ and the set $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a frame for a subspace of $\mathrm{Sym}^k(V)$. Denoting the frame operator associated with $X^{(k)}$ by $\mathcal{F}^{(k)}$, note that

$$\mathrm{tr} \ \mathcal{F}^{(k)} = \sum_{i=1}^{m} \langle x_i^{\otimes k}, x_i^{\otimes k} \rangle = \sum_{i=1}^{m} \langle x_i, x_i \rangle^k = m \ .$$

Thus applying inequality (4) of Lemma 2.1 to $\mathcal{F}^{(k)}$ gives

$$||\mathcal{F}^{(k)}||^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}}$$

as desired. $\square$

In the above derivation, the binomial coefficient in the denominator of the Welch bounds has an explicit geometric interpretation as the dimension of the subspace on which $\mathcal{G}^{\circ k}$ operates non-trivially.

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[2]The authors have recently become aware of a manuscript by S. Waldron, in preparation for publication as a book chapter [31], that presents a similar perspective to the one set forth in this section.

[3]In fact, it has been shown in [19] that, almost always, $\mathrm{rank}(\mathcal{G}^{\circ k}) = \min(\binom{n+k-1}{k}, m)$.

As already pointed out in Section 1.1, the Welch bounds given by (1) can be obtained as a corollary to Theorem 2.3.

Corollary 2.4. *Suppose that $\{x_1, \ldots, x_m\}$ are unit vectors in $\mathbb{H}$ that span a subspace V of dimension n. Let $c_{\max} = \max_{i \neq j} |\langle x_i, x_j \rangle|$. Then for all integers $k \geqslant 1$*

$$c_{\max}^{2k} \geqslant \frac{1}{m-1} \left[\frac{m}{\binom{n+k-1}{k}} - 1 \right].$$

Proof. Due to Theorem 2.3, (6) holds and is equivalent to

$$\sum_{i \neq j} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{m^2}{\binom{n+k-1}{k}} - m.$$

Because the $m(m-1)$ terms in the sum on the left are all non-negative, their maximum must be at least as large as their average and the result follows. □

3. Tightness of the Welch bounds

As noted in Section 1.1, several authors have investigated conditions under which the Welch bound with $k = 1$ is satisfied with equality. A condition for all $k \geqslant 1$ is given below, followed by some discussion about when these higher-order Welch bounds are meaningful. However, sets that attain the bounds are hard to construct. This naturally leads to the notion of *Approximate Welch Bound Equality* (AWBE) sets, which is also addressed in this section.

As prevalent in the literature [10, 35, 17, 23, 18, 27, 32], a set X that meets inequality (2) with equality is known as a *Welch Bound Equality (WBE)* set. If X meets inequality (1) with equality, it is called a *Maximal Welch Bound Equality (MWBE)* set [10, 35, 23, 27].

3.1. Conditions for Equality

Conditions for equality to hold in the Cauchy-Schwarz inequality imply that equality holds in (4) of Lemma 2.1 if and only if all the eigenvalues of T are equal. This is used in the following.

Theorem 3.1 (Equality in the Welch Bounds). *Given a set of unit vectors $X = \{x_1, \ldots, x_m\}$ in $\mathbb{H}$ that span an n-dimensional subspace V, let $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$. Then for integer $k \geq 1$,*

(i) $X^{(k)}$ is a WBE set if and only if $X^{(k)}$ is a tight frame for the space $\mathrm{Sym}^k(V)$.

(ii) $X^{(k)}$ is an MWBE set if and only if $X^{(k)}$ is an equiangular tight frame for the space $\mathrm{Sym}^k(V)$.

In both cases the frame bound is $\frac{m}{\binom{n+k-1}{k}}$.

Proof. (i) When $k = 1$, having a WBE set is equivalent to having equality in (5) of Theorem 2.2. Due to Lemma 2.1, this means that that all the non-zero eigenvalues of the Grammian $\mathcal{G}$ (and of the frame operator $\mathcal{F}$) associated with X must be equal to m/n. This holds if and only if X is a tight frame for V, in which case $\mathcal{F} = \frac{m}{n}\mathcal{I}_V$.

When $k \geqslant 1$, having a WBE set is equivalent to having equality in (6) of Theorem 2.3. The $m \times m$ Gram matrix associated with the set $X^{(k)}$ is

$$
\mathcal{G}_{X^{(k)}} = \begin{bmatrix} \langle x_1^{\otimes k}, x_1^{\otimes k} \rangle & \cdots & \langle x_1^{\otimes k}, x_m^{\otimes k} \rangle \\ \vdots & \ddots & \vdots \\ \langle x_m^{\otimes k}, x_1^{\otimes k} \rangle & \cdots & \langle x_m^{\otimes k}, x_m^{\otimes k} \rangle \end{bmatrix} = \begin{bmatrix} \langle x_1, x_1 \rangle^k & \cdots & \langle x_1, x_m \rangle^k \\ \vdots & \ddots & \vdots \\ \langle x_m, x_1 \rangle^k & \cdots & \langle x_m, x_m \rangle^k \end{bmatrix}
$$

which is the same as the k-fold Hadamard product of $\mathcal{G}$, i.e, $\mathcal{G}^{\circ k}$. From Lemma 2.1, equality holds if and only if all the non-zero eigenvalues of $\mathcal{G}_{X^{(k)}}$ are equal to $\frac{m}{\binom{n+k-1}{k}}$. This is the same as the eigenvalues of the frame operator $\mathcal{F}^{(k)}$. The set $X^{(k)}$ is therefore a WBE set if and only if it is a tight frame for $\mathrm{Sym}^k(V)$ with frame operator

$$
\mathcal{F}^{(k)} = \frac{m}{\binom{n+k-1}{k}}\mathcal{I}_{\mathrm{Sym}^k(V)} \; .
$$

(ii) Using the fact that the maximum in a set of non-negative numbers is greater than or equal to the average and the result of Theorem 2.3, one gets

$$
\max_{i \neq j} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{1}{m(m-1)} \sum_{i \neq j} |\langle x_i, x_j \rangle|^{2k} \geqslant \frac{1}{m-1} \left[\frac{m}{\binom{n+k-1}{k}} - 1 \right] \; .
$$

Thus, to be a MWBE set, equality must hold in both the inequalities. In part (i) it has been established that equality holds in the second inequality if and only

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if $X^{(k)}$ is a tight frame for $\mathrm{Sym}^k(V)$. Equality holds in the first inequality if and only if $|\langle x_i, x_j \rangle|$ is constant for all $i \neq j$, i.e., if and only if the vectors in X are *equiangular* in $\mathbb{H}$. Due to the inner product property (3), this means that $X^{(k)}$ must be an equiangular set in $\mathrm{Sym}^k(V)$, when $k \geq 1$. The set $X^{(k)}$ is therefore a MWBE set if and only if it is an equiangular tight frame for $\mathrm{Sym}^k(V)$. The frame bound $\frac{m}{\binom{n+k-1}{k}}$ comes from part (i).

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.2. Non-triviality of the bounds

A necessary condition for (2) to not be vacuous is that

$$m > \binom{n+k-1}{k}.$$

For a fixed n, this suggests that $m > \mathcal{O}(n^k)$ as $k \to \infty$, thereby implying that for higher values of k one can hope for meaningful bounds only when $m \gg n$. Similarly, if k is fixed, useful bounds require $m > \max(n, \binom{n+k-1}{k})$. This implies that $m > \mathcal{O}(k^{n-1})$ as $n \to \infty$. In any case, it is evident that the bounds for large k are only significant when $m \gg n$.

3.3. Approximate Welch bound equality sequences

Pairs (m, n) for which equiangular tight frames of m vectors in $\mathbb{C}^n$ can exist along with the required conditions and examples are given in [28, 3, 14, 2, 29]. Due to Theorem 3.1, these are MWBE sets for $k = 1$. If MWBE sets do not exist for a certain pair (m, n) when $k = 1$, then MWBE sets of size m also cannot exist for $k > 1$. This is because, by the inner product property (3), for two pairs (i, j) and (i', j'),

$$|\langle x_i, x_j \rangle_{\mathbb{C}^n}| \neq |\langle x_{i'}, x_{j'} \rangle_{\mathbb{C}^n}|$$

implies

$$\left| \langle x_i^{\otimes k}, x_j^{\otimes k} \rangle_{\mathrm{Sym}^k(\mathbb{C}^n)} \right| \neq \left| \langle x_{i'}^{\otimes k}, x_{j'}^{\otimes k} \rangle_{\mathrm{Sym}^k(\mathbb{C}^n)} \right|.$$

There are not many values of k for which MWBE sets can be constructed. The maximum number of equiangular lines in $\mathbb{C}^n$ is n^2 [8, 30]. Due to (3),

equiangular lines in $\text{Sym}^k(\mathbb{C}^n)$ are also equiangular lines in $\mathbb{C}^n$ and so a necessary condition for the existence of MWBE sets is

$$\binom{n+k-1}{k} \le n^2. \tag{7}$$

For a fixed dimension n, there are not many values of k that satisfy (7); in fact, for $n \geqslant 3$ there is no $k > 2$ for which MWBE sets can exist.

However, for $k \le 2$ and $m = n^2$, MWBE sets are the same as symmetric informationally complete positive operator-valued measures (SIC-POVMs), which have been studied extensively in connection to quantum measurement [21, 17, 22]. SIC-POVMs are hard to construct [17, 21]; existence of SIC-POVMs in all dimensions n has been conjectured [36, 21, 24].

In the context of sets that are *not* MWBE sets, for $k = 1$, the author in [33] addresses the construction of equiangular frames for $\mathbb{R}^n$ that are *not* tight and comments on the potential application of these in signal processing and quantum information theory.

When $k \le 2$, mutually unbiased bases (MUBs) give rise to WBE sets of $n(n+1)$ elements in $\mathbb{C}^n$ (Theorem 3, [17]) but these are challenging to construct [17]. Theorem 3.1 characterizes WBE sets, for $k \ge 1$, in terms of tight frames for the space of symmetric k-tensors. These sets are also equivalent to complex projective k-designs (Theorem 2, [21] and Theorem 1, [17]). However, complex projective k-designs and hence WBE sets (for $k > 2$) are hard to find and known not to exist in many cases [22]. Consequently, it seems reasonable to look for sets that are as close as possible to attaining the bound in (2) for a given k.

Definition 3.2 (Approximate Welch Bound Equality Sets)**.** *Let V be an n-dimensional subspace of a Hilbert space $\mathbb{H}$. For $k \ge 1$, if a set $X = \{x_1, \ldots, x_m\}$ of unit vectors in V minimizes $\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i^{\otimes k}, x_j^{\otimes k} \rangle|^2$ then $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is called an* Approximate *Welch Bound Equality (AWBE) set.*

Definition 3.2 is inspired by the notion of the frame potential and its connection to tight frames in $\mathbb{C}^n$ [1]. The quantity $\sum_{i=1}^{m} \sum_{j=1}^{m} |\langle x_i^{\otimes k}, x_j^{\otimes k} \rangle|^2$ is the frame potential of the set $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ in $\text{Sym}^k(\mathbb{C}^n)$. When $k = 1$,

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the minimizer of the frame potential is a tight frame for $\mathbb{C}^n$ (Theorem 7.1, [1]) and hence a WBE set.

Example 3.3 (Approximate WBE sets). To illustrate numerically the construction of an AWBE set via minimization of the frame potential, consider the case with $k = 3$, $n = 2$ and $m = 7$. To construct an AWBE set with these parameters, start with a frame $X_0 = \{x_1, \ldots, x_7\}$ of seven unit vectors in $\mathbb{C}^2$, such as

$$X_0 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right\}.$$

Noting that $\sum_{i=1}^7 \sum_{j=1}^7 |\langle x_i^{\otimes 3}, x_j^{\otimes 3} \rangle|^2 = \sum_{i=1}^7 \sum_{j=1}^7 |\langle x_i, x_j \rangle|^6$ and using X_0 as the initial point, solve

$$\min \sum_{i=1}^7 \sum_{j=1}^7 |\langle x_i, x_j \rangle|^6$$

subject to the constraint that the vectors x_i in the solution are unit normed. The solution, obtained by Matlab, is the set

$$X = \begin{bmatrix} 0.99 & 0.14 & 0.56 & -0.68 & 0.93 & -0.86 & -0.30 \\ 0.08 & 0.99 & 0.83 & 0.73 & -0.36 & -0.50 & 0.95 \end{bmatrix}.$$

The ratio of the upper and lower bounds for the frame obtained from $X^{(3)}$ in $\mathrm{Sym}^3(\mathbb{C}^2)$ (i.e., the condition number of the frame operator) is 3. Although not tight, $X^{(3)}$ is as close to tight, in the sense of frame potential energy discussed above, as any frame of seven unit vectors that are pure tensors in the space $\mathrm{Sym}^3(\mathbb{C}^2)$ can be. $X^{(3)}$ is an AWBE set of seven vectors.

It is noteworthy that the set X has lower and upper frame bound equal to 3.5. Within the numerical precision of this example, X is thus a tight frame for $\mathbb{C}^2$, and the AWBE set $X^{(3)}$ arises as the tensor powers of the elements of a tight frame. Indeed, this phenomenon has been observed consistently by the authors in numerous numerical experiments of this type. Tight frames in V minimize $\sum_i \sum_j |\langle x_i, x_j \rangle|^2$ while AWBE sets in $\mathrm{Sym}^k(V)$ minimize $\sum_i \sum_j |\langle x_i^{\otimes k}, x_j^{\otimes k} \rangle|^2 = \sum_i \sum_j |\langle x_i, x_j \rangle|^{2k}$.

4. Connection to Sampling of Homogeneous Polynomials and Compressed Sensing

It is well known (see, e.g., [26]) that $H_{(0,k)}$, the linear space of homogeneous polynomials of total degree k in variables $\bar{z}^{(1)}, \ldots, \bar{z}^{(n)}$ is isomorphic to $\mathrm{Sym}^k(V)$. This section, in Theorem 4.1 below, points out a connection between the condition that a set is a frame for $\mathrm{Sym}^k(V)$ and the reconstructability of polynomials in $H_{(0,k)}$ from the values they take at sets of m points in $\mathbb{C}^n$.

Theorem 4.1. *Let $X = \{x_1, \ldots, x_m\}$ be a set of vectors in V. For $k \geqslant 1$, polynomials in $H_{(0,k)}$ are uniquely determined from their samples at the points in X if and only if $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a frame for $\mathrm{Sym}^k(V)$.*

Proof. (i) Let $k = 1$ and $w \in V = \mathrm{Sym}^1(V)$. Denote by $[w^{(1)} \cdots w^{(n)}]^{\mathsf{T}}$ the coordinates of w in some orthonormal basis for V. The mapping $p : V \to H_{(0,1)}$ defined by

$$p(w) = w^{(1)} \bar{z}^{(1)} + \cdots w^{(n)} \bar{z}^{(n)} = p_w(z^{(1)}, \ldots, z^{(n)})$$

is an isomorphism that takes $w \in V$ to the polynomial $p_w \in H_{(0,1)}$.

If X is a frame for V, the associated Bessel map $F : V \to \mathbb{C}^m$ is given by

$$F(w) = \begin{bmatrix} \langle x_1, w \rangle \\ \vdots \\ \langle x_m, w \rangle \end{bmatrix} = \begin{bmatrix} p_w(x_1^{(1)}, \ldots, x_1^{(n)}) \\ \vdots \\ p_w(x_m^{(1)}, \ldots, x_m^{(n)}) \end{bmatrix}. \tag{8}$$

$F(w)$ is a vector of values obtained by evaluating or sampling p_w at $x_1, \ldots, x_m$. Define a sampling function $P_X : H_{(0,1)} \to \mathbb{C}^m$ by

$$P_X(p) = \begin{bmatrix} p(x_1^{(1)}, \ldots, x_1^{(n)}) \\ \vdots \\ p(x_m^{(1)}, \ldots, x_m^{(n)}) \end{bmatrix}.$$

Note that (8) shows that the Bessel map is given by $F(w) = P_X(p_w)$. Because F is invertible, w is *uniquely* determined by $F(w)$. Hence any $p_w \in H_{(0,1)}$ is uniquely determined by its samples $P_X(p_w)$.

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Conversely, if X does not form a frame for V, the mapping F defined by (8) has a non-trivial kernel K. In this case, $P_X(p_w) = P_X(p_{w+u})$ for all $u \in K$. Therefore, p_w is not uniquely determined from its samples at $x_1, ..., x_m$.

(ii) For $k > 1$, the space of interest is $\mathrm{Sym}^k(V)$ and the frame is $X^{(k)}$. As for $k = 1$, mapping a polynomial to its coefficient sequence defines an embedding of $H_{(0,k)}$ in $\mathrm{Sym}^k(V)$. If $v = w^{\otimes k} \in \mathrm{Sym}^k(V)$ is a pure tensor power of $w \in V$, then

$$F^{(k)}(v) = \begin{bmatrix} \langle x_1^{\otimes k}, w^{\otimes k} \rangle \\ \vdots \\ \langle x_m^{\otimes k}, w^{\otimes k} \rangle \end{bmatrix} = \begin{bmatrix} \langle x_1, w \rangle^k \\ \vdots \\ \langle x_m, w \rangle^k \end{bmatrix} = \begin{bmatrix} p_v(x_1) \\ \vdots \\ p_v(x_m) \end{bmatrix}$$

where $p_v \in H_{(0,k)}$ defined by $p_v(z) = \langle z, w \rangle^k$. $\mathrm{Sym}^k(V)$ is spanned by pure tensor powers of elements in V [26]. Thus, for arbitrary $v \in \mathrm{Sym}^k(V)$, $F^{(k)}(v)$ is a vector of m samples of a polynomial in $H_{(0,k)}$ taken at points $x_1, ..., x_m$. Thus, a polynomial $p \in H_{(0,k)}$ is uniquely determined by the sample set

$$P_X^{(k)}(p) = \begin{bmatrix} p(x_1) \\ \vdots \\ p(x_m) \end{bmatrix}$$

if and only if $X^{(k)}$ is a frame for $\mathrm{Sym}^k(V)$. $\qquad\square$

Remark 4.2 (*Connection to compressed sensing*). A signal $x \in \mathbb{C}^N$ is k-sparse in a basis $\Psi = \{\psi_j\}_{j=1}^N$ if x is a weighted superposition of at most k elements of Ψ. Compressed sensing is broadly concerned with the inverse problem of reconstructing such a signal x from linear measurements $\{y_\ell = \langle x, \phi_\ell \rangle \mid \ell = 1, \ldots, n\}$ with $n \ll N$. In the general setting, one has $\Phi x = y$, where Φ is a $n \times N$ sensing matrix having the measurement vectors ϕ_ℓ as its rows, x is a length-N signal and y is a length-n measurement.

The standard compressed sensing technique guarantees exact recovery of the signal x with high probability if Φ satisfies the Restricted Isometry Property (RIP) [4, 5, 12]. This means that for a fixed k, there exists a small number δ_k, such that

$$(1 - \delta_k)\|x\|_{\ell_2}^2 \leq \|\Phi x\|_{\ell_2}^2 \leq (1 + \delta_k)\|x\|_{\ell_2}^2,$$

for any k-sparse signal x. Denoting any $n \times k$ submatrix of Φ by Φ_T, the above is equivalent to saying that all the eigenvalues of $\Phi_T^* \Phi_T$ must lie in $[1 - \delta_k, 1 + \delta_k]$, or, that the rows of Φ_T form a frame with frame bounds very close to each other and to 1. From the characterization of WBE sets given in Theorem 3.1, this means that if the rows of Φ_T is a WBE set of n vectors in $\mathbb{C}^k$ then Φ is a "good" sensing matrix.

5. Generalized Frames

Let $\mathbb{H}$ be a complex Hilbert space and (M, μ) a measure space. A *generalized frame* in $\mathbb{H}$ indexed by M is a family of vectors $\{x_\alpha \in \mathbb{H} : \alpha \in M\}$, denoted by (X_M, μ) or just X_M, such that:

(a) For every $y \in \mathbb{H}$, the function $\tilde{y} : M \to \mathbb{C}$ defined by

$$\tilde{y}(\alpha) = \langle x_\alpha, y \rangle_{\mathbb{H}}$$

is μ-measurable.

(b) There exist constants $0 < A \leqslant B < \infty$ such that, for every $y \in \mathbb{H}$,

$$A||y||_{\mathbb{H}}^2 \leqslant \int_M |\langle x_\alpha, y \rangle_{\mathbb{H}}|^2 d\mu(\alpha) \leqslant B||y||_{\mathbb{H}}^2$$

or,

$$A||y||_{\mathbb{H}}^2 \leqslant ||\tilde{y}||_{L^2(M,\mu)}^2 \leqslant B||y||_{\mathbb{H}}^2 .$$

The mapping $F : \mathbb{H} \to L^2(M, \mu)$ is given by $F(y) = \langle x_\alpha, y \rangle \Big|_{\alpha \in M}$ and its adjoint is $F^* : L^2(M, \mu) \to \mathbb{H}$ with $F^*(g) = \int_M g(\alpha) x_\alpha d\mu(\alpha)$. The frame operator $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is $\mathcal{F} = F^* F$; i.e., for $y \in \mathbb{H}$

$$\mathcal{F}(y) = \int_M \langle x_\alpha, y \rangle x_\alpha \, d\mu(\alpha) .$$

The Grammian $\mathcal{G} : L^2(M, \mu) \to L^2(M, \mu)$ is defined by $\mathcal{G} = F F^*$; i.e.,

$$(\mathcal{G}f)(\beta) = \int_M \langle x_\beta, x_\alpha \rangle f(\alpha) \, d\mu(\alpha) .$$

A good overview of generalized frames is given in [16].

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5.1. Welch bounds for generalized frames

With V an n-dimensional subspace of $\mathbb{H}$, denote by S^{n-1} the set of unit vectors in V. For each $x \in S^{n-1}$, the mapping $\Pi_x : V \to \text{span}(x)$ given by

$$\Pi_x(v) = \langle x, v \rangle\, x$$

is a projector that maps V onto the one-dimensional subspace spanned by x. Since $\Pi_x = \Pi_{e^{i\theta}x}$ for any $\theta \in [0, 2\pi)$, the collection of projectors Π_x is parameterized by the complex projective space $\mathbb{CP}^{n-1}$. Given a normalized measure μ on $\mathbb{CP}^{n-1}$ (i.e., with $\mu(\mathbb{CP}^{n-1}) = 1$), a generalized frame $X_{\mathbb{CP}^{n-1}}$ for V is obtained by selecting one representative $x \in \mathbb{H}$ from each equivalence class corresponding to a point in $\mathbb{CP}^{n-1}$. The frame operator $\mathcal{F}_\mu : V \to V$ for this generalized frame is given by

$$\mathcal{F}_\mu(v) = \int_{\mathbb{CP}^{n-1}} \Pi_x(v) d\mu(x) = \int_{\mathbb{CP}^{n-1}} \langle x, v \rangle\, x\; d\mu(x)\;.$$

Theorem 5.1 (First Welch Bound)**.** *Let μ be a normalized measure on $\mathbb{CP}^{n-1}$ and $X_{\mathbb{CP}^{n-1}}$ be a generalized frame for an n-dimensional subspace V of a Hilbert space $\mathbb{H}$. Then*

$$\iint_{\mathbb{CP}^{n-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \geqslant \frac{1}{n},$$

with equality if and only if the frame is tight.

Proof. Taking $\{e_1, \ldots, e_n\}$ to be an orthonormal basis of V, the trace of $\mathcal{F}_\mu$ is given by

$$\text{tr}\,\mathcal{F}_\mu = \sum_{k=1}^{n} \langle \mathcal{F}_\mu(e_k), e_k \rangle = 1.$$

The Hilbert-Schmidt norm of $\mathcal{F}_\mu$ is

$$||\mathcal{F}_\mu||_V^2 = \text{tr}\,\mathcal{F}_\mu^* \mathcal{F}_\mu = \iint_{\mathbb{CP}^{n-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y).$$

Using Lemma 2.1 in this setting for $k = 1$ gives

$$\iint_{\mathbb{CP}^{n-1}} |\langle x, y \rangle|^2 d\mu(x) d\mu(y) \geqslant \frac{1}{n}\;.$$

The bound is achieved if and only if

$$\mathcal{F}_\mu = \frac{1}{n} \mathcal{I}_V,$$

i.e., if and only if the generalized frame is tight. □

For $k \geqslant 1$, higher-order Welch bounds in the generalized frame setting are obtained by considering $\mathrm{Sym}^k(V)$. The projector $\Pi_{x^{\otimes k}}$ maps $\mathrm{Sym}^k(V)$ onto the one-dimensional subspace spanned by the tensor power $x^{\otimes k}$, with $x \in S^{n-1}$. Direct calculation using (3) yields

$$\Pi_{x^{\otimes k}} = \Pi_x^{\otimes k},$$

and, for $v \in V$,

$$\Pi_x^{\otimes k} v^{\otimes k} = \langle x, v \rangle^k x^{\otimes k} .$$

This collection of projectors is parameterized by $\mathbb{CP}^{n-1}$. Corresponding to each $x \in \mathbb{CP}^{n-1}$, choosing a representative unit vector in V yields a collection of unit vectors

$$X_{\mathbb{CP}^{n-1}}^{(k)} = \{ u_x^{\otimes k} \mid u_x \in V, x \in \mathbb{CP}^{n-1} \}.$$

Given a normalized measure μ on $\mathbb{CP}^{n-1}$, $X_{\mathbb{CP}^{n-1}}^{(k)}$ becomes a generalized frame for $\mathrm{Sym}^k(V)$ with frame operator $\mathcal{F}_\mu^{(k)} : \mathrm{Sym}^k(V) \to \mathrm{Sym}^k(V)$ given by

$$\mathcal{F}_\mu^{(k)} = \int_{\mathbb{CP}^{n-1}} \Pi_{x^{\otimes k}} d\mu(x) .$$

Theorem 5.2 (Higher Welch Bounds)**.** *Let μ be a normalized measure on $\mathbb{CP}^{n-1}$ and let $X_{\mathbb{CP}^{n-1}}$ be a generalized frame for an n-dimensional subspace V of a Hilbert space $\mathbb{H}$. Then for all $k \geq 1$,*

$$\iint_{\mathbb{CP}^{n-1}} | \langle x, y \rangle |^{2k} d\mu(x) d\mu(y) \geqslant \frac{1}{\binom{n+k-1}{k}}, \tag{9}$$

with equality if and only if $(X_{\mathbb{CP}^{n-1}}^{(k)}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^k(V)$.

Proof. Noting that $\mathrm{tr}\, \mathcal{F}_\mu^{(k)} = 1$, Lemma 2.1 implies

$$||\mathcal{F}_\mu^{(k)}||_{\mathrm{Sym}^k(V)}^2 = \iint_{\mathbb{CP}^{n-1}} | \langle x, y \rangle |^{2k} d\mu(x) d\mu(y) \geqslant \frac{1}{\binom{n+k-1}{k}}$$

with equality if and only if

$$\mathcal{F}_\mu^{(k)} = \frac{1}{\binom{n+k-1}{k}} \mathcal{I}_{\mathrm{Sym}^k(V)},$$

i.e., if and only if $(X_{\mathbb{CP}^{n-1}}^{(k)}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^k(V)$. □

Example 5.3. Let $X = \{x_1, \ldots, x_m\}$ be a set of unit vectors that is a frame for an n-dimensional subspace V of $\mathbb{H}$. Consider the (normalized) discrete measure

$$\mu = \frac{1}{m} \sum_{x \in X} \delta_x \ .$$

Using this measure in Theorem 5.2 yields

$$\frac{1}{m^2} \sum_{x,y \in X} |\langle x, y \rangle|^{2k} \geqslant \frac{1}{\binom{n+k-1}{k}}$$

which is equivalent to (2). Equality is obtained if and only if $X^{(k)} = \{x^{\otimes k} | \ x \in X\}$ is a tight frame for $\mathrm{Sym}^k(V)$, i.e., if and only if

$$\frac{1}{m} \sum_{x \in X} \Pi_{x^{\otimes k}} = \frac{1}{\binom{n+k-1}{k}} \mathcal{I}_{\mathrm{Sym}^k(V)} \ .$$

Thus the generalized frame perspective yields the Welch bound for finite frames as a special case.

Example 5.4. If $X = \{x_i\}_{i=1}^{\infty}$ in $\mathbb{CP}^{n-1}$ and $\{w_i\}_{i=1}^{\infty}$ is a summable set of positive numbers, defining a discrete measure by

$$\mu = \frac{\sum_{i=1}^{\infty} w_i \delta_{x_i}}{\sum_{j=1}^{\infty} w_j}$$

yields a generalized frame (X, μ). With this measure in Theorem 5.2, one gets

$$\frac{1}{\left(\sum_{j=1}^{\infty} w_j \right)^2} \sum_{i,\ell} |\langle w_i x_i, w_\ell x_\ell \rangle|^{2k} \geqslant \frac{1}{\binom{n+k-1}{k}} \ .$$

Thus the generalized frame perspective also produces Welch bounds for countably infinite frames.

5.2. Tight generalized frames and Haar measure

Roy and Scott [26] discuss relationships between Haar measure on the n-dimensional unitary group $\mathcal{U}_n$, the unique unitarily invariant probability measure it induces on $\mathbb{CP}^{n-1}$, and unweighted t-designs. The formulation in the preceding section enables a frame-theoretic perspective on this circle of ideas.

$\mathcal{U}_n$ acts transitively on $\mathbb{CP}^{n-1}$ and for each $y \in V^{\otimes k}$ define

$$\Phi_U^{(k)}(y) = U^{\otimes k} y$$

where $U \in \mathcal{U}_n$ [20]. $\mathrm{Sym}^k(V)$ is an invariant subspace of $V^{\otimes k}$ under this action.

Equality in (9) is attained if and only if $(X_{\mathbb{CP}^{n-1}}^{(k)}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^k(V)$. It is now shown that this occurs when μ is the Haar measure.

Theorem 5.5. *Let μ be the normalized $\mathcal{U}_n$-invariant Haar measure on $\mathbb{CP}^{n-1}$. Then, for all $k \geqslant 1$, $(X_{\mathbb{CP}^{n-1}}^{(k)}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^k(V)$, i.e.,*

$$\mathcal{F}_\mu^{(k)} = \frac{1}{\binom{n+k-1}{k}} \mathcal{I}_{\mathrm{Sym}^k(V)}.$$

Proof. For any $U \in \mathcal{U}_n$,

$$\begin{aligned}
U^{\otimes k} \mathcal{F}_\mu^{(k)} (U^{\otimes k})^* &= \int_{\mathbb{CP}^{n-1}} U^{\otimes k} \Pi_x^{\otimes k} (U^{\otimes k})^* \, d\mu(x) \\
&= \int_{\mathbb{CP}^{n-1}} \Pi_{Ux}^{\otimes k} \, d\mu(x) \\
&= \int_{\mathbb{CP}^{n-1}} \Pi_x^{\otimes k} \, d\mu(x)
\end{aligned}$$

where the last equality holds because μ is $\mathcal{U}_n$-invariant. This shows that $\mathcal{F}_\mu^{(k)}$ commutes with all $U^{\otimes k}$. Since $\Phi_U^{(k)}$ acts on $\mathrm{Sym}^k(V)$ irreducibly, Schur's lemma implies $\mathcal{F}_\mu^{(k)} = \lambda \mathcal{I}_{\mathrm{Sym}^k(V)}$. Because $\mathrm{tr}\, \mathcal{F}_\mu^{(k)} = 1$, dimensionality considerations imply that

$$\lambda = \frac{1}{\dim \mathrm{Sym}^k(V)} = \frac{1}{\binom{n+k-1}{k}} \ .$$

$\square$

5.3. Homogeneous polynomials and t-designs

As in the finite frame case, the generalized frame perspective yields connections to homogeneous polynomials and, further, to spherical t-designs. Suppose that $(X_{\mathbb{CP}^{n-1}}^{(k)}, \mu)$ is a generalized tight frame for $\mathrm{Sym}^k(V)$. Then

$$\mathcal{F}_\mu^{(k)} = \int_{\mathbb{CP}^{n-1}} \Pi_{x^{\otimes k}} d\mu(x) = \frac{1}{\binom{n+k-1}{k}} \mathcal{I}_{\mathrm{Sym}^k(V)} \ . \tag{10}$$

The mapping $F^{(k)} : \operatorname{Sym}^k(V) \to L^2(\mathbb{CP}^{n-1}, \mu)$ is given by $F^{(k)}(w) = \langle x^{\otimes k}, w \rangle |_{x \in \mathbb{CP}^{n-1}}$ for $w \in \operatorname{Sym}^k(V)$. Since the tensor powers $\{v^{\otimes k} : v \in V\}$ span $\operatorname{Sym}^k(V)$, $\langle x^{\otimes k}, w \rangle$ can be written as a linear combination of terms of the form $\langle x, w \rangle^k$ and hence is in $H_{(0,k)}$. Denoting this polynomial associated with w by p_w and using (10) gives, for any $v, w \in \operatorname{Sym}^k(V)$,

$$
\begin{aligned}
\langle v, w \rangle &= \binom{n+k-1}{k} \int_{\mathbb{CP}^{n-1}} \langle v, x^{\otimes k} \rangle \langle x^{\otimes k}, w \rangle \, d\mu(x) \\
&= \binom{n+k-1}{k} \int_{\mathbb{CP}^{n-1}} \overline{p_v(x)} p_w(x) \, d\mu(x) .
\end{aligned}
$$

If μ is the normalized discrete measure discussed in Example 5.3 the frame operator $\mathcal{F}_\mu^{(k)}$ can be written as

$$
\frac{1}{m} \sum_{x \in X} \Pi_{x^{\otimes k}} = \frac{1}{\binom{n+k-1}{k}} \mathcal{I}_{\operatorname{Sym}^k(V)} .
$$

Using this representation of $\mathcal{I}_{\operatorname{Sym}^k(V)}$ gives

$$
\langle v, w \rangle = \frac{\binom{n+k-1}{k}}{m} \sum_{x \in X} \overline{p_v(x)} p_w(x)
$$

so that

$$
\int_{\mathbb{CP}^{n-1}} \overline{p_v(x)} p_w(x) \, d\mu(x) = \frac{1}{m} \sum_{x \in X} \overline{p_v(x)} p_w(x) .
$$

This implies that, for any $g \in H_{(k,k)}$, the space of homogeneous polynomials of total degree k in $x_1, \ldots, x_m$ and total degree k in $\bar{x}_1, \ldots, \bar{x}_m$,

$$
\int_{\mathbb{CP}^{n-1}} g(x) \, d\mu(x) = \frac{1}{m} \sum_{x \in X} g(x) .
$$

If $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ is a tight frame for $\operatorname{Sym}^k(V)$ for all $k \leqslant t$, then

$$
\int_{\mathbb{CP}^{n-1}} g(x) \, d\mu(x) = \frac{1}{m} \sum_{x \in X} g(x) \tag{11}
$$

for all $g \in \bigoplus_{k=1}^t H_{(k,k)}$. Equation (11) defines $X^{(k)} = \{x_1^{\otimes k}, \ldots, x_m^{\otimes k}\}$ as a complex projective t-design [17].

6. Conclusion

The classical Welch bounds have been shown to arise from dimensionality considerations in connection with frame and Grammian operators. Geometric derivations of the first Welch bound have been given in previous work. This paper has extended the geometric perspective to obtain the higher-order Welch bounds, with the kth bound for $k \geqslant 1$ arising naturally from observing either the k-fold Hadamard product of the Grammian or the frame operator associated with a frame on a space of symmetric k-tensors.

Welch bounds for generalized frames have been derived and the classical case shown to follow from this more general result. The role of tight frames in achieving the Welch bounds with equality has been established in this general setting. In general, for $k \geq 2$, due to the difficulty in construction and, in some cases, lack of existence of Welch bound equality sets, it is natural to construct sets that come as close as possible to attaining the bound. This was done here by considering sets that minimize the frame potential energy in the space $\text{Sym}^k(V)$ of symmetric k-tensors, and has led to an open question regarding conditions under which such sets must arise from tight frames of V. Further, specific connections have been clarified between the circle of ideas entailed in the geometric understanding of the Welch bounds and related topics involving symmetric tensors, homogeneous polynomials, and t-designs. In particular, it has been shown that a homogeneous polynomial of a known degree k can be uniquely reconstructed from its samples, provided the sampling points form a frame for the space of symmetric k-tensors.

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References

[1] J. J. Benedetto and M. Fickus. Finite normalized tight frames. *Advances in Computational Mathematics*, 18:357–385, 2003.

[2] B. G. Bodmann and H. J. Elwood. Complex equiangular Parseval frames and Seidel matrices containing pth roots of unity. *Proc. Amer. Math. Soc.*, 138(12):4387–4404, 2010.

[3] B. G. Bodmann, V. I. Paulsen, and M. Tomforde. Equiangular tight frames from complex Seidel matrices containing cube roots of unity. *Linear Algebra Appl.*, 430(1):396–417, 2009.

[4] E. Candès. Compressive sampling. In *Proceedings of the International Congress of Mathematicians*, volume III, pages 1433–1452. European Mathematical Society, Madrid, 2006.

[5] E. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. *IEEE Transactions on Information Theory*, 52(2):489 – 509, February 2006.

[6] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2003.

[7] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.

[8] P. Delsarte, J. M. Goethals, and J. J. Seidel. Bounds for systems of lines and Jacobi polynomials. *Philips Res. Repts.*, 30(3):91–105, 1975. Issue in honour of C.J. Bouwkamp.

[9] P. Delsarte and V. Levenshtein. Association schemes and coding theory. *IEEE Transactions on Information Theory*, 44:2477–2504, 1998.

[10] C. Ding and T. Feng. A generic construction of complex codebooks meeting the Welch bound. *IEEE Transactions on Information Theory*, 53(11), November 2007.

[11] C. T. J. Dodson and T. Poston. *Tensor Geometry: The Geometric Viewpoint and its Uses*, volume 130 of *Graduate Texts in Mathematics*. Springer, second edition, 1991.

[12] D. L. Donoho. Compressed sensing. *IEEE Transactions on Information Theory*, 52(4):1289 – 1306, April 2006.

[13] S. G. Hoggar. *t*-designs in projective spaces. *European J. Combin.*, 3(3):233–254, 1982.

[14] R. B. Holmes and V. I. Paulsen. Optimal frames for erasures. *Linear Algebra Appl.*, 377:31–51, 2004.

[15] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.

[16] G. Kaiser. *A Friendly Guide to Wavelets*. Birkhäuser, Boston, 1994.

[17] A. Klappenecker and M. Rötteler. Mutually unbiased bases are complex projective 2-designs. In *Proceedings of the International Symposium on Information Theory*, pages 1740 – 1744, September 2005.

[18] J. L. Massey and T. Mittelholzer. Welch's bound and sequence sets for code-division multiple-access systems. In R. Capocelli, A. De Santis, and U. Vaccaro, editors, *Sequences II: Methods in Communication, Security and Computer Science*, pages 63–78. Springer-Verlag, New York, 1991.

[19] I. Peng and S. Waldron. Signed frames and Hadamard products of Gram matrices. *Linear Algebra and its Applications*, 347:131–157, 2002.

[20] A. M. Perelomov. *Generalized coherent states and their applications*. Springer-Verlag, Berlin, 1986.

[21] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves. Symmetric informationally complete quantum measurements. *Journal of Mathematical Physics*, 45(6):2171 – 2180, 2004.

[22] A. Roy and A. J. Scott. Weighted complex projective 2-designs from bases: Optimal state determination by orthogonal measurements. *Journal of Mathematical Physics*, 48(072110), 2007.

[23] D. V. Sarwate. Meeting the Welch bound with equality. In T. Helleseth and H. Niederreiter, editors, *Sequences and Their Applications*, DMTCS Series, chapter 11. Springer-Verlag, 1999.

[24] A. J. Scott and M. Grassl. Symmetric informationally complete positive operator-valued measure: A new computer study. *Journal of Mathematical Physics*, 51(4), 2010.

[25] H. Shapiro, 1998. Personal communication.

[26] R. Shaw. *Linear Algebra and Group Representations*, volume 2. Academic Press, 1983.

[27] T. Strohmer and R. W. Heath, Jr. Grassmannian frames with applications to coding and communication. *Applied and Computational Harmonic Analysis*, 14(3):257–275, 2003.

[28] M. A. Sustik, J. A. Tropp, I. S. Dhillon, and R. W. Heath, Jr. On the existence of equiangular tight frames. *Linear Algebra Appl.*, 426(2-3):619–635, 2007.

[29] J. A. Tropp. Complex equiangular tight frames. In *Proc. SPIE Wavelets XI*, pages 590412.01–11, 2005.

[30] J. H. van Lint and J. J. Seidel. Equilateral point sets in elliptic geometry. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:335–348, 1966.

[31] S. Waldron. *An Introduction to Finite Tight Frames*. Applied and Numerical Harmonic Analysis. Birkhäuser. To appear.

[32] S. Waldron. Generalized Welch bound equality sequences are tight frames. *IEEE Transactions on Information Theory*, 49(9):2307–2309, 2003.

[33] S. Waldron. On the construction of equiangular tight frames from graphs. *Linear Algebra and its Applications*, 431(11):2228–2242, 2009.

[34] L. R. Welch. Lower bounds on the maximum cross correlation of signals. *IEEE Transactions on Information Theory*, 20(3):397–399, May 1974.

[35] P. Xia, S. Zhou, and G. B. Giannakis. Achieving the Welch bound with difference sets. *IEEE Transactions on Information Theory*, 51(5), May 2005.

[36] G. Zauner. *Quantendesigns - Grundzüge einer nichtkommutativen Designtheorie*. PhD thesis, Universität Wien, Vienna, Austria, 1999. Available online: http://www.mat.univie.ac.at/ neum/papers/physpapers.html.