

Application of Zernike polynomials in solving certain first and second order partial differential equations

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Abstract

Integration operational matrix methods based on Zernike polynomials are used to determine approximate solutions of a class of non-homogeneous partial differential equations (PDEs) of first and second order. Due to the nature of the Zernike polynomials being described in the unit disk, this method is particularly effective in solving PDEs over a circular region. Further, the proposed method can solve PDEs with discontinuous Dirichlet and Neumann boundary conditions, and as these discontinuous functions cannot be defined at some of the Chebyshev or Gauss-Lobatto points, the much acclaimed pseudo-spectral methods are not directly applicable to such problems. Solving such PDEs is also a new application of Zernike polynomials as so far the main application of these polynomials seem to have been in the study of optical aberrations of circularly symmetric optical systems. In the present method, the given PDE is converted to a system of linear equations of the form $A\mathbf{x} = \mathbf{b}$ which may be solved by both l_1 and l_2 minimization methods among which the l_1 method is found to be more accurate. Finally, in the expansion of a function in terms of Zernike polynomials, the rate of decay of the coefficients is given for certain classes of functions.

Keywords: integration operational matrix, Laplace equation, partial differential equations, Zernike polynomials

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1 Introduction

1.1 Background

If the analytical solution of a partial differential equation (PDE) with a forcing function and given boundary conditions is difficult, we go for numerical methods as discussed in [4], [18], and [19]. In many practical cases such as fluid flow in a rotating cylinder, electromagnetic equations in cylindrical waveguides and optical lens design, and many other cases of a cylindrical or a spherical boundary with axial symmetry, one needs to solve PDEs over a disk instead of a rectangular boundary [4, 5, 18].

One of the important numerical methods to solve PDEs is using integration operational matrices (IOMs), first introduced in [9] who showed that integral and differential equations could be reduced

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to a set of linear algebraic equations with an approximation in the sense of least-squares by taking an orthonormal set of Walsh functions. This approach was subsequently applied to solve PDEs in *rectangular* regions using piecewise constant orthogonal functions (PCOF) and orthogonal polynomials (OP), comprehensive accounts of which are available in [23] and [10]. A set of 2-D orthogonal functions known as Zernike polynomials and defined on the unit disk is used in the analysis and evaluation of optical systems with circular pupils by expanding optical wavefront functions in series of these polynomials [28, 20, 3]. It appears that these polynomials have never been used in the analysis of PDEs and in this paper they are fruitfully employed to solve PDEs on a disk.

With the above motivation, the main contribution of this paper is a new method of solving PDEs in circular regions with discontinuous Dirichlet and Neumann boundary conditions using Zernike polynomials and IOMs. In practice, to get an IOM, a pre-selected set of orthogonal functions (OFs) is first integrated analytically. The result of integration is then expressed in terms of a fixed finite number of functions in the original set of OFs. This gives an approximation of the integration operator. On the other hand, for a set of OPs, a three-term derivative recurrence relation is available, which on integration allows one to express any orthogonal function in the set in terms of the original set of OPs. For the radial parts of Zernike polynomials, neither a three-term recurrence relation is available, nor on integrating any radial polynomial in the set can it be expressed automatically in terms of other polynomials in the set. However, this difficulty can be obviated by using a derivative relation of the radial parts of Zernike polynomials given in [21], different from the three-term derivative recurrence relation of OPs, which needs a trivial matrix inversion to get the IOM of the radial parts of Zernike polynomials.

If a known function $u(r, \theta)$ is approximated by the Zernike polynomials $R_n^m(r)e^{im\theta}$ of order (m, n) , m being the azimuthal frequency and n the degree of the radial polynomial $R_n^m(r)$, then the radial polynomials of degree higher than n are neglected. In the method proposed here, to obtain the approximate solution of a PDE, these higher order polynomials are approximated by lower order polynomials with some reliable interpolation formula such as Lagrange's, see Remark 2.1, equations (3.7) and (3.8). If this projection of the neglected polynomials on the space generated by the lower order polynomials is not done then our IOM method of solving PDEs using Zernike polynomials may fail. In [23], the author used two dimensional block pulse functions to solve second order PDEs. However, the approximate solutions did not converge, and this may be attributed to the fact that the above mentioned idea of projecting higher order terms was not considered.

Using IOMs, a given PDE is reduced to a system of linear equations of the form $\mathbf{Ax} = \mathbf{b}$, where A is a sparse matrix. This must then be solved to obtain an approximate solution of the PDE. This is usually solved with least squares approximation by using standard matrix pseudo-inverse or Moore-Penrose inverse and is called l_2 method. An alternative method to solve such a sparse system is an l_1 -minimization algorithm developed in [6] which is used in this paper and found to be more accurate than the least squares solution.

As a comparison with some of the existing methods, it may be noted that pseudo-spectral collocation methods seem unsuited for solving PDEs with discontinuous boundary conditions. In any of the pseudo-spectral collocation methods, the Chebyshev points:

$$\frac{1}{2}(\cos \frac{i\pi}{M} + 1), \quad i = 0, 1, \dots, M$$

or the Gauss-Lobatto points in the interval $[0,1]$ are chosen so as to minimize Runge phenomenon, and the boundary conditions (BCs) must be defined at these discrete points. In the problem that we have considered here in Example 3.1 of Section 3, this would imply that the mixed BCs are discontinuous for $i = M$. Therefore, the given boundary conditions will not be defined at one of the collocation points.

The method proposed here can naturally take care of such discontinuous BCs although Gibbs-Wilbraham phenomenon will still appear. This is expected when using Fourier series, and cosine and sine functions are part of the structure of Zernike polynomials.

One of the primary concerns in numerically solving PDEs is the convergence of the solutions. The ingenious basis functions developed by Livermore et al. in [17] to apply Galerkin's method on the disk and sphere behave asymptotically as Jacobi polynomials for large degree, and so their convergence rates are similar to the latter polynomials. In the polynomial representation of a function $f(x)$ of one variable, where the domain is an interval, it is generally true that Chebyshev polynomials give the fastest rate of convergence from the larger family of Jacobi polynomials except when $f(x)$ is singular at one or both end points. In this setting, all Gegenbauer polynomials (including Legendre and Chebyshev) converge equally fast at the endpoints, but Gegenbauer polynomials converge more rapidly on the interior with increasing order of the degree m . However, for functions on the unit disk, Zernike polynomials are superior in terms of rate of convergence when compared to Chebyshev-Fourier series [5]. So, in terms of the rate of convergence, the performance of the method proposed in this paper using the IOM of Zernike polynomials has the same superiority as discussed in [5]. Moreover, it is also shown in this paper (see Theorem 4.3) that for functions that are Hölder continuous of order λ , the coefficients C_{nm} in the expansion of a function $u(r, \theta)$ in terms of Zernike polynomials $R_n^m(r)e^{im\theta}$ decay at least like $m^{-\lambda+1}$, $\lambda \geq 1$. Using methods similar to the decay of Fourier coefficients for functions of a single variable, a similar result (Theorem 4.1) is given for functions that are k times continuously differentiable. Some preliminary ideas underlying the Zernike polynomials are given next.

1.2 Preliminaries and Notation

Zernike polynomials are used to conveniently expand optical wavefront functions that arise in optical systems with circular pupils [28, 3, 16]. Proposed by F. von Zernike in [28], these polynomials are orthogonal on the unit disk $B(0, 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and can be found in the following way [28, 20, 3]. To start with, one considers a partial differential equation that is invariant under rotations of the coordinate axes about the origin. Such an equation has the form:

$$\Delta U + \alpha \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 U + \beta \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) U + \gamma U = 0. \quad (1.1)$$

In polar coordinates r and ϕ , by using the transformation: $x = r \cos \phi$ and $y = r \sin \phi$, equation (1.1) becomes

$$(1 + \alpha r^2) \frac{\partial^2 U}{\partial r^2} + \left(\frac{1}{r} + (\alpha + \beta)r \right) \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \gamma U = 0. \quad (1.2)$$

Choosing $\alpha = -1$, $\beta = -1$, and $\gamma = n(n+2)$ gives the hypergeometric equation

$$x(1-x) \frac{d^2 y}{dx^2} + (1-2x) \frac{dy}{dx} + \frac{1}{4} \left[n(n+2) - \frac{m^2}{x} \right] y = 0. \quad (1.3)$$

The solution of (1.3) denoted by $R_n^m(r)$ is known as the *radial part of a Zernike polynomial* and is given by (1.4) below when n, m are non-negative integers, and $n - m$ is even and non-negative, see [20], [22] and [3]:

$$R_n^m(r) = \sum_{\ell=0}^{\frac{n-m}{2}} (-1)^\ell \frac{(n-\ell)!}{\ell! \left(\frac{n-m}{2} - \ell\right)! \left(\frac{n+m}{2} - \ell\right)!} r^{n-2\ell}. \quad (1.4)$$

The radial part of Zernike polynomials can be expressed in terms of classical Jacobi polynomials defined on the interval $[0, 1]$, as outlined in [27] and [11]. The classical Jacobi polynomials satisfy a three term recurrence relation, a second order differential equation as (1.3), and interesting properties such as (2.9), see [1] and [8]. A method of computing the radial parts of Zernike polynomials of arbitrary degree using the discrete Fourier transform has been discussed in [15]. In [24], a recurrence relation that depends neither on the degree nor on the azimuthal order of the radial polynomials is developed. The Zernike polynomials are solutions to (1.1) or (1.2), and are given by

$$U_n^m(r, \phi) = R_n^m(r)(C_1 \cos m\phi + C_2 \sin m\phi), \quad n \in \mathbb{N} \cup \{0\}, \quad n - m \geq 0, \quad n - m \text{ even}, \quad (1.5)$$

where C_1 and C_2 are arbitrary constants.

It is worth mentioning that *Zernike polynomials* is the general name for a class of bivariate orthogonal polynomials on the unit disk, and they are a particular case of orthogonal polynomials on the unit disk, see [11]. They are defined by a radial part that is a univariate orthogonal Jacobi polynomial defined on the interval $[0, 1]$, and a non-radial part that is a bivariate spherical harmonic. The Zernike polynomials form a complete orthogonal set for the interior of the unit disk $B(0, 1)$, see [2, 20, 3]. They can therefore be normalized to form an orthonormal basis for the space $L^2(B(0, 1))$, see Section 4. Let $f(r, \phi)$ be an arbitrary function defined on $B(0, 1)$. In terms of the Zernike polynomials given in (1.5), f can be represented as [28, 3]

$$f(r, \phi) = \sum_{n=0}^{\infty} \sum_{\substack{0 \leq m \leq n \\ n-m \text{ even}}} (A_{nm} \cos m\phi + B_{nm} \sin m\phi) R_n^m(r) \quad (1.6)$$

where

$$A_{nm} = \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi) \cos m\phi R_n^m(r) r d\phi dr, \quad B_{nm} = \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi) \sin m\phi R_n^m(r) r d\phi dr, \quad (1.7)$$

and ϵ_m is the Neumann factor:

$$\epsilon_m = \begin{cases} 1 & \text{if } m = 0, \\ 2 & \text{otherwise.} \end{cases}$$

An efficient algorithm for calculating the coefficients A_{nm} and B_{nm} is discussed in [22], see also [14]. An approximation of f of order (m, n) can then be calculated as

$$f(r, \phi) = \sum_{i=0}^n \sum_{\substack{0 \leq j \leq i \\ i-j \text{ even}}}^m (A_{ij} \cos j\phi + B_{ij} \sin j\phi) R_i^j(r).$$

For details on the properties of Zernike polynomials the reader is referred to [20] and [3]. In the context of opto-mechanical analysis by finite element methods, a Zernike polynomial representation of the surface distortions is found to be better than by other orthogonal functions [12].

At times it will be convenient to write a product of matrices in a vector and tensor product form. In such cases, a matrix P of size $m \times n$ is represented as a vector of size mn , and written as $\text{vec}(P)$. The tensor product of matrices is denoted by \otimes . To clarify, let A, B, X , and Y be 2×2 matrices and

consider

$$\begin{aligned}
& Y = AXB \\
\text{or, } & \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \\
& \text{or, } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{21} & a_{12}b_{21} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{21} & a_{22}b_{21} \\ a_{11}b_{12} & a_{12}b_{12} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{12} & a_{22}b_{12} & a_{21}b_{22} & a_{22}b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\
& \implies \text{vec}(Y) = (B^T \otimes A)\text{vec}(X) \tag{1.8}
\end{aligned}$$

The above method of transforming a linear system of matrix unknowns to a linear system involving a vector of unknowns by means of tensor product (also known as Kronecker product) is well known (Chapter 4; [13]).

1.3 Outline

The remaining part of the paper is organized as follows. Section 2 discusses the solution of a first order PDE by using the IOM method with Zernike polynomials, and the accuracy of the method is shown by means of a specific example. Section 3 discusses the solution of a second order PDE by using the IOM method with Zernike polynomials. The results of the proposed method are applied to a particular Laplace equation. Surface plots of the solutions and error estimates are provided for various orders of approximation. Section 4 provides some results related to the decay of the Zernike coefficients of functions that are k times continuously differentiable and ones that are Hölder continuous. Some derivations in obtaining the IOMs are given in Section 5, and Section 6 has some concluding remarks including future directions.

2 Solving first order partial differential equations using integration operational matrix

A linear first order partial differential equation (FOPDE) in $u(x, y)$ with forcing function $f(x, y)$ has the general form

$$\alpha(x, y) \frac{\partial u}{\partial x} + \beta(x, y) \frac{\partial u}{\partial y} + \gamma(x, y)u = f(x, y). \tag{2.1}$$

In general, getting the analytical solution of (2.1), subject to some boundary conditions, is often not feasible or too cumbersome. Consequently, one seeks numerical methods to solve such problems. This section describes such a technique using the integrational operational matrix (IOM) of Zernike polynomials. The technique shown below can be adapted for some given α, β, γ . For the sake of demonstration, we consider the following form of a FOPDE:

$$\alpha x \frac{\partial u}{\partial x} + \beta y \frac{\partial u}{\partial y} + \gamma u = f, \tag{2.2}$$

where α, β , and γ are constants. Changing (2.2) to polar coordinates (r, ϕ) gives

$$r(\alpha \cos^2 \phi + \beta \sin^2 \phi) \frac{\partial u}{\partial r} - (\alpha - \beta) \sin \phi \cos \phi \frac{\partial u}{\partial \phi} + \gamma u = f. \tag{2.3}$$

Equation (2.3) has to be solved subject to the boundary conditions

$$\begin{aligned} u(r_0, \phi) &= h(\phi), \\ u(r, \phi_0) &= g(r). \end{aligned}$$

Integrating (2.3) first with respect to r from r_0 to r and then with respect to ϕ from ϕ_0 to ϕ using integration by parts and the given boundary conditions gives

$$\begin{aligned} & \alpha r \int_{\phi_0}^{\phi} u \cos^2 \phi \, d\phi + \beta r \int_{\phi_0}^{\phi} u \sin^2 \phi \, d\phi - r_0 \int_{\phi_0}^{\phi} h(\phi)(\alpha \cos^2 \phi + \beta \sin^2 \phi) \, d\phi - \\ & - \int_{\phi_0}^{\phi} (\alpha \cos^2 \phi + \beta \sin^2 \phi) \left[\int_{r_0}^r u \, dr \right] \, d\phi - \frac{(\alpha - \beta)}{2} \left[\int_{r_0}^r \sin 2\phi \, u(r, \phi) - \sin 2\phi_0 \int_{r_0}^r g(r) \, dr \right. \\ & \left. - 2 \int_{r_0}^r \int_{\phi_0}^{\phi} u \cos 2\phi \, d\phi \, dr \right] + \gamma \int_{\phi_0}^{\phi} \int_{r_0}^r u \, dr \, d\phi = \int_{\phi_0}^{\phi} \int_{r_0}^r f \, dr \, d\phi. \end{aligned} \quad (2.4)$$

To solve for u , matrix representations for the integral operators, the forcing function f , and the unknown u , in terms of trigonometric and radial parts of Zernike polynomials, are needed. The idea is to write every term in (2.4) in terms of an integration operational matrix and reduce (2.4) to an algebraic equation.

Let the trigonometric functions be written as a vector

$$\Phi(\phi) = [1, \cos \phi, \sin \phi, \cos 2\phi, \sin 2\phi, \dots]^T.$$

For practical purposes, only a finite number of terms of Φ can be used. If only terms up to azimuthal frequency $m\phi$ are used, then, by an abuse of notation, we shall also denote by Φ the resulting vector of size $M = 2m + 1$, i.e.,

$$\Phi(\phi) = [1, \cos \phi, \sin \phi, \cos 2\phi, \sin 2\phi, \dots, \cos m\phi, \sin m\phi]^T,$$

and

$$\begin{aligned} \int_{\phi_0}^{\phi} \Phi(\phi) \, d\phi &= [\phi - \phi_0, \sin \phi - \sin \phi_0, -\cos \phi + \cos \phi_0, \frac{\sin 2\phi}{2} - \frac{\sin 2\phi_0}{2}, \\ & \dots, \frac{\sin m\phi}{m} - \frac{\sin m\phi_0}{m}, -\frac{\cos m\phi}{m} + \frac{\cos m\phi_0}{m}]^T. \end{aligned}$$

In order to express the above integral in matrix form, the ϕ appearing on the right side has to be expressed in terms of $\{1, \cos \phi, \sin \phi, \cos 2\phi, \sin 2\phi, \dots\}$. To achieve this, we take the Fourier series expansion of ϕ over $[0, 2\pi]$ which is

$$\phi = \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin k\phi.$$

This yields

$$\int_{\phi_0}^{\phi} \Phi(\phi) d\phi = \begin{bmatrix} \pi - \phi_0 & 0 & -2 & 0 & -1 & 0 & -\frac{2}{3} & \dots & 0 & -\frac{2}{m} \\ -\sin \phi_0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \cos \phi_0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -\frac{\sin 2\phi_0}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ \frac{\cos 2\phi_0}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -\frac{\sin m\phi_0}{m} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{m} \\ \frac{\cos m\phi_0}{m} & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos \phi \\ \sin \phi \\ \cos 2\phi \\ \sin 2\phi \\ \vdots \\ \cos m\phi \\ \sin m\phi \end{bmatrix} = E_{\phi\phi_0} \Phi.$$

The radial parts of Zernike polynomials $R_n^m(r)$ can be written sequentially as an infinite vector as

$$R(r) = [R_0^0(r), R_1^1(r), R_2^0(r), R_2^2(r), R_3^1(r), R_3^3(r), \dots, R_n^m(r), \dots]^T, \quad n \in \mathbb{N} \cup \{0\}, \quad 0 \leq n - m, \quad n - m \text{ even}.$$

For a fixed n , the number of radial polynomials of degree less than or equal to n is denoted by N . Thus, for approximation with radial polynomials with degree up to n , only N elements of the above vector $R(r)$ are used. As in the case of Φ , by an abuse of notation, this is also denoted by $R(r)$ and is given by

$$R(r) = [R_0^0(r), R_1^1(r), R_2^0(r), R_2^2(r), R_3^1(r), R_3^3(r), \dots, R_n^m(r)]^T, \quad n \in \mathbb{N} \cup \{0\}, \quad 0 \leq n - m, \quad n - m \text{ even}.$$

The solution u represented in terms of the Zernike polynomials then results in an approximation

$$\tilde{u} = \Phi^T U R, \quad (2.5)$$

where U contains the coefficients of u with respect to the polynomials up to a chosen degree. To help in understanding, first consider the first term on the left side of (2.4). This can be written as

$$\alpha r \int_{\phi_0}^{\phi} \cos^2 \phi \Phi^T U R d\phi. \quad (2.6)$$

Take $\phi_0 = 0$. Then

$$\int_0^{\phi} \cos^2 \phi \Phi(\phi) d\phi = \begin{bmatrix} \frac{\pi}{2} & 0 & -1 & 0 & -\frac{1}{4} & 0 & -\frac{1}{3} & 0 & -\frac{1}{4} & 0 & -\frac{1}{5} & \dots & 0 & \frac{-1}{m-2} & 0 & \frac{-1}{m-1} & 0 & \frac{-1}{m} \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \pi/4 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{16} & 0 & -\frac{1}{10} & \dots & 0 & \frac{-1}{2(m-2)} & 0 & -\frac{1}{2(m-1)} & 0 & \frac{-1}{2m} \\ \frac{5}{16} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{20} & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{15} & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{20} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{4(m-2)} & 0 & 0 & 0 & \frac{1}{2m} \\ x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \frac{-1}{4(m-2)} & 0 & 0 & 0 & \frac{-1}{2m} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos \phi \\ \sin \phi \\ \cos 2\phi \\ \sin 2\phi \\ \cos 3\phi \\ \sin 3\phi \\ \cos 4\phi \\ \sin 4\phi \\ \cos 5\phi \\ \sin 5\phi \\ \vdots \\ \cos m\phi \\ \sin m\phi \end{bmatrix}$$

$$= E_{\phi}^{\cos^2 \phi} \Phi(\phi),$$

where $x_1 = \frac{1}{2m} + \frac{1}{4(m-2)} + \frac{1}{4(m+2)}$ in the last row of the matrix,

$$\int_0^\phi \sin^2 \phi \Phi(\phi) d\phi =$$

$$\begin{bmatrix} \frac{\pi}{2} & 0 & -1 & 0 & -\frac{3}{4} & 0 & -\frac{1}{3} & 0 & -\frac{1}{4} & 0 & -\frac{1}{5} & \cdots & 0 & \frac{-1}{m-2} & 0 & \frac{-1}{m-1} & 0 & \frac{-1}{m} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{3}{4} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 & \frac{1}{8} & 0 & \frac{1}{10} & \cdots & 0 & \frac{1}{2(m-2)} & 0 & \frac{1}{2(m-1)} & 0 & \frac{1}{2m} \\ \frac{3}{16} & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & -\frac{1}{20} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2}{15} & \frac{1}{4} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & \frac{1}{20} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(m-2)} & 0 & 0 & 0 & \frac{1}{2m} \\ x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{4(m-2)} & 0 & 0 & 0 & \frac{-1}{2m} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos \phi \\ \sin \phi \\ \cos 2\phi \\ \sin 2\phi \\ \cos 3\phi \\ \sin 3\phi \\ \cos 4\phi \\ \sin 4\phi \\ \cos 5\phi \\ \sin 5\phi \\ \vdots \\ \cos m\phi \\ \sin m\phi \end{bmatrix}$$

$$= E_\phi^{\sin^2 \phi} \Phi(\phi),$$

where $x_2 = \frac{1}{2m} - \frac{1}{4(m-2)} - \frac{1}{4(m+2)}$ in the last row of the matrix, and

$$\int_0^\phi \cos 2\phi \Phi(\phi) d\phi =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{3} & 0 & -\frac{1}{8} & 0 & 0 & \cdots & 0 & -\frac{1}{m-2} & 0 & -\frac{1}{m-1} & 0 & -\frac{1}{m} \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2(m-2)} & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2(m-2)} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos \phi \\ \sin \phi \\ \cos 2\phi \\ \sin 2\phi \\ \cos 3\phi \\ \sin 3\phi \\ \cos 4\phi \\ \sin 4\phi \\ \cos 5\phi \\ \sin 5\phi \\ \vdots \\ \cos m\phi \\ \sin m\phi \end{bmatrix} = E_\phi^{\cos 2\phi} \Phi(\phi).$$

where $x_3 = \frac{1}{2(m+2)} + \frac{1}{2(m-2)}$ in the last row of the matrix.

When one considers integrating from some angle ϕ_0 to ϕ instead of from zero to ϕ then the following adjustments have to be made. The first column of $E_\phi^{\cos^2 \phi}$ changes to

$$\begin{bmatrix} \frac{\pi}{2} - \frac{\phi_0}{2} - \frac{\sin 2\phi_0}{4} \\ -\frac{3}{4} \sin \phi_0 - \frac{\sin 3\phi_0}{12} \\ \frac{\cos \phi_0}{4} + \frac{\cos 3\phi_0}{12} \\ \vdots \\ -\frac{\sin(m-2)\phi_0}{4(m-2)} - \frac{\sin m\phi_0}{2m} - \frac{\sin(m+2)\phi_0}{4(m+2)} \\ \frac{\cos(m-2)\phi_0}{4(m-2)} + \frac{\cos m\phi_0}{2m} + \frac{\cos(m+2)\phi_0}{4(m+2)} \end{bmatrix},$$

and the resulting matrix will be denoted by $E_{\phi\phi_0}^{\cos^2\phi}$.[†] The first column of $E_{\phi}^{\sin^2\phi}$ changes to

$$\begin{bmatrix} \frac{\pi}{2} - \frac{\phi_0}{2} + \frac{\sin 2\phi_0}{4} \\ -\frac{1}{4} \sin \phi_0 + \frac{\sin 3\phi_0}{12} \\ \frac{3}{4} \cos \phi_0 - \frac{\cos 3\phi_0}{12} \\ \vdots \\ \frac{\sin(m-2)\phi_0}{4(m-2)} - \frac{\sin m\phi_0}{2m} + \frac{\sin(m+2)\phi_0}{4(m+2)} \\ -\frac{\cos(m-2)\phi_0}{4(m-2)} + \frac{\cos m\phi_0}{2m} - \frac{\cos(m+2)\phi_0}{4(m+2)} \end{bmatrix},$$

and the resulting matrix will be denoted by $E_{\phi\phi_0}^{\sin^2\phi}$. The first column of $E_{\phi}^{\cos 2\phi}$ becomes

$$\begin{bmatrix} -\frac{\sin 2\phi_0}{2} \\ -\frac{\sin \phi_0}{2} - \frac{\sin 3\phi_0}{6} \\ -\frac{\cos \phi_0}{2} + \frac{\cos 3\phi_0}{6} \\ -\frac{\phi_0}{2} - \frac{\sin 4\phi_0}{8} \\ \frac{\pi}{2} - \frac{\phi_0}{2} - \frac{\sin 4\phi_0}{8} \\ \frac{\cos 4\phi_0}{8} \\ \vdots \\ -\frac{\sin(m-2)\phi_0}{2(m-2)} - \frac{\sin(m+2)\phi_0}{2(m+2)} \\ \frac{\cos(m-2)\phi_0}{2(m-2)} + \frac{\cos(m+2)\phi_0}{2(m+2)} \end{bmatrix},$$

and the resulting matrix will be denoted by $E_{\phi\phi_0}^{\cos 2\phi}$. For demonstration purposes, to keep things less cumbersome, the radial parts of Zernike polynomials up to degree three will be used below, and will be denoted by the vector $R(r)$.

$$R(r) = [R_0^0(r), R_1^1(r), R_2^0(r), R_2^2(r), R_3^1(r), R_3^3(r)]^T = [1, r, 2r^2 - 1, r^2, 3r^3 - 2r, r^3]^T. \quad (2.7)$$

In (2.6), $rR(r)$ can be approximated as

$$rR(r) = \begin{bmatrix} r \\ r^2 \\ 2r^3 - r \\ r^3 \\ 3r^4 - 2r^2 \\ r^4 \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ r \\ 2r^2 - 1 \\ r^2 \\ 3r^3 - 2r \\ r^3 \end{bmatrix} = M_R^r R(r). \quad (2.8)$$

In (2.8), the higher order terms involving r^4 are ignored making the last row of M_R^r equal to zero. See Remark 2.1 for how to get a better approximation by projecting terms involving r^4 on the space generated by $R(r)$. The first two terms of (2.4) can thus be written as[‡]

$$\alpha r \int_{\phi_0}^{\phi} u \cos^2 \phi \, d\phi + \beta r \int_{\phi_0}^{\phi} u \sin^2 \phi \, d\phi = \alpha \Phi^T \left(E_{\phi\phi_0}^{\cos^2\phi} \right)^T U M_R^r R + \beta \Phi^T \left(E_{\phi\phi_0}^{\sin^2\phi} \right)^T U M_R^r R.$$

[†]In general, an integration operational matrix will be denoted by E , the subscript indicating the limits of integration while the superscript indicating the integrand.

[‡]Strictly speaking, the resulting matrix representation is an approximation of some integral. However, we will always use the equality sign to express that the operators on the right side of the equality stand for corresponding integrals on the left side.

The function $h(\phi)$ is expanded in terms of trigonometric functions as

$$h(\phi) = \mathbf{h}^\top \Phi(\phi) = \Phi(\phi)^\top \mathbf{h},$$

where \mathbf{h} contains the coefficients of h in terms of the functions in Φ . Also, r_0 can be expressed as

$$r_0 = M_{r_0}^\top R(r),$$

where M_{r_0} is a vector whose first entry is r_0 and the rest are zero. These expressions for $h(\phi)$ and r_0 transform the third term in (2.4) to

$$r_0 \int_{\phi_0}^{\phi} h(\phi) (\alpha \cos^2 \phi + \beta \sin^2 \phi) d\phi = \alpha \Phi^\top \left(E_{\phi\phi_0}^{\cos^2 \phi} \right)^\top \mathbf{h} M_{r_0}^\top R(r) + \beta \Phi^\top \left(E_{\phi\phi_0}^{\sin^2 \phi} \right)^\top \mathbf{h} M_{r_0}^\top R(r).$$

Using the recurrence relation given in [21] and [22]

$$\int_{r_0}^r [R_n^m(r) + R_n^{m+2}(r)] dr = \frac{1}{n+1} [R_{n+1}^{m+1}(r) - R_{n-1}^{m+1}(r)] \Big|_{r_0}^r, \quad (2.9)$$

gives

$$\int_{r_0}^r R(\rho) d\rho \approx \begin{bmatrix} -R_1^1(r_0) & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}R_2^2(r_0) & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{3}[R_3^1(r_0) - R_1^1(r_0) - R_3^3(r_0)] & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3}R_3^3(r_0) & 0 & 0 & 0 & 0 & \frac{1}{3} \\ -\frac{1}{4}[R_4^2(r_0) - R_2^2(r_0) - R_4^4(r_0)] & 0 & 0 & -1 & 0 & 0 \\ -\frac{1}{4}R_4^4(r_0) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} R(r) = E_{rr_0} R(r), \quad (2.10)$$

where recall that $R_0^0(r) = 1$, $R_1^1(r) = r$, $R_2^0(r) = 2r^2 - 1$, $R_2^2(r) = r^2$, $R_3^1(r) = 3r^3 - 2r$, and $R_3^3(r) = r^3$. Once again, as in (2.8), the higher order terms involving r^4 are ignored resulting in zeros in the last row of E_{rr_0} in (2.10), see Remark 2.1. Even though the derivation here uses only radial parts of Zernike polynomials up to degree three, the general expression of the integration operational matrix E_{rr_0} , using all radial polynomials up to some given degree n , is given in Section 5. Using (2.10), the fourth term of (2.4) is

$$\begin{aligned} \alpha \int_{\phi_0}^{\phi} \cos^2 \phi \left[\int_{r_0}^r u dr \right] d\phi &= \alpha \int_{\phi_0}^{\phi} \cos^2 \phi \int_{r_0}^r \Phi^\top U R dr d\phi = \alpha \int_{\phi_0}^{\phi} \cos^2 \phi \Phi^\top d\phi U \int_{r_0}^r R dr \\ &= \alpha \left(E_{\phi\phi_0}^{\cos^2 \phi} \Phi \right)^\top U E_{rr_0} R, \end{aligned}$$

and similarly

$$\beta \int_{\phi_0}^{\phi} \sin^2 \phi \left[\int_{r_0}^r u dr \right] d\phi = \beta \left(E_{\phi\phi_0}^{\sin^2 \phi} \Phi \right)^\top U E_{rr_0} R.$$

If $\sin 2\phi \Phi$ is written as

$$\sin 2\phi \Phi = M_{\Phi}^{\sin 2\phi} \Phi$$

then

$$\frac{\alpha - \beta}{2} \int_{r_0}^r \sin 2\phi u(r, \phi) dr = \int_{r_0}^r \sin 2\phi \Phi^\top U R dr = \Phi^\top \left(M_{\Phi}^{\sin 2\phi} \right)^\top U E_r R.$$

Expressing $g(r)$ as

$$g(r) = \mathbf{g}^T R(r)$$

gives

$$\int_{r_0}^r g(r) dr = \int_{r_0}^r \mathbf{g}^T R(r) dr = \mathbf{g}^T E_{rr_0} R(r).$$

Also, $\sin 2\phi_0$ can be expressed as

$$\sin 2\phi_0 = M_{\sin 2\phi_0}^T \Phi(\phi) = \Phi(\phi)^T M_{\sin 2\phi_0},$$

where $M_{\sin 2\phi_0}$ is a vector whose first entry is $\sin 2\phi_0$ and the rest are zero. Therefore,

$$\frac{\alpha - \beta}{2} \sin 2\phi_0 \int_{r_0}^r g(r) dr = \frac{\alpha - \beta}{2} \Phi^T M_{\sin 2\phi_0} \mathbf{g}^T E_{rr_0} R(r).$$

In matrix form,

$$\int_{\phi_0}^{\phi} \cos 2\phi \Phi = E_{\phi\phi_0}^{\cos 2\phi} \Phi,$$

and thus

$$\frac{\alpha - \beta}{2} \cdot 2 \int_{r_0}^r \int_{\phi_0}^{\phi} u \cos 2\phi d\phi dr = (\alpha - \beta) \Phi^T \left(E_{\phi\phi_0}^{\cos 2\phi} \right)^T U E_{rr_0} R.$$

The last term on the left side of (2.4) becomes, in matrix form,

$$\begin{aligned} \gamma \int_{\phi_0}^{\phi} \int_{r_0}^r u dr d\phi &= \gamma \int_{\phi_0}^{\phi} \int_{r_0}^r \Phi^T U R dr d\phi = \gamma \left(\int_{\phi_0}^{\phi} \Phi^T d\phi \right) U \left(\int_{r_0}^r R dr \right) \\ &= \gamma (E_{\phi\phi_0} \Phi)^T U E_{rr_0} R. \end{aligned}$$

Finally, expressing the forcing function f in terms of the Zernike polynomials as

$$f = \Phi^T F R,$$

where F contains the coefficients of f , the integral on the right side of (2.4) can be written as

$$\int_{\phi_0}^{\phi} \int_{r_0}^r f dr d\phi = (E_{\phi\phi_0} \Phi)^T F E_{rr_0} R.$$

The operator matrices and the corresponding notation are summarized below:

1. $\int_{\phi_0}^{\phi} \Phi d\phi = E_{\phi\phi_0} \Phi.$
2. $\int_{\phi_0}^{\phi} \cos^2 \phi \Phi d\phi = E_{\phi\phi_0}^{\cos^2 \phi} \Phi.$
3. $\int_{\phi_0}^{\phi} \sin^2 \phi \Phi d\phi = E_{\phi\phi_0}^{\sin^2 \phi} \Phi.$
4. $\int_{r_0}^r R(r) dr = E_{rr_0} R(r).$

$$5. \int_{\phi_0}^{\phi} \cos 2\phi \Phi d\phi = E_{\phi\phi_0}^{\cos 2\phi} \Phi.$$

Putting everything together, (2.4) reduces to the algebraic equation

$$\begin{aligned} \Phi^T & \left[\alpha \left(E_{\phi\phi_0}^{\cos^2 \phi} \right)^T U M_R^r + \beta \left(E_{\phi\phi_0}^{\sin^2 \phi} \right)^T U M_R^r - \alpha \left(E_{\phi\phi_0}^{\cos^2 \phi} \right)^T \mathbf{h} M_{r_0}^T - \beta \left(E_{\phi\phi_0}^{\sin^2 \phi} \right)^T \mathbf{h} M_{r_0}^T \right. \\ & - \alpha \left(E_{\phi\phi_0}^{\cos^2 \phi} \right)^T U E_{rr_0} - \beta \left(E_{\phi\phi_0}^{\sin^2 \phi} \right)^T U E_{rr_0} - \frac{(\alpha - \beta)}{2} \left(M_{\Phi}^{\sin 2\phi} \right)^T U E_{rr_0} + \frac{\alpha - \beta}{2} M_{\sin 2\phi_0} \mathbf{g}^T E_{rr_0} \\ & \left. + (\alpha - \beta) \left(E_{\phi\phi_0}^{\cos 2\phi} \right)^T U E_{rr_0} + \gamma E_{\phi\phi_0}^T U E_{rr_0} \right] R(r) = \Phi^T E_{\phi\phi_0}^T F E_{rr_0} R(r). \end{aligned} \quad (2.11)$$

Equation (2.11) has to be solved for the matrix U to get an approximation of the solution u of the original PDE (2.3). To solve for U , it is convenient to rewrite (2.11) using the vector and tensor product representation introduced in (1.8) of Section 1. With this notation, (2.11) becomes

$$\begin{aligned} & \left(\alpha M_R^{r \ T} \otimes \left(E_{\phi\phi_0}^{\cos 2\phi} \right)^T + \beta M_R^{r \ T} \otimes \left(E_{\phi\phi_0}^{\sin 2\phi} \right)^T - \alpha E_{rr_0}^T \otimes \left(E_{\phi\phi_0}^{\cos 2\phi} \right)^T - \beta E_{rr_0}^T \otimes \left(E_{\phi\phi_0}^{\sin 2\phi} \right)^T \right. \\ & \left. - \frac{\alpha - \beta}{2} E_{rr_0}^T \otimes \left(M_{\Phi}^{\sin 2\phi} \right)^T + (\alpha - \beta) E_{rr_0}^T \otimes \left(E_{\phi\phi_0}^{\cos 2\phi} \right)^T + \gamma E_{rr_0}^T \otimes E_{\phi\phi_0}^T \right) \text{vec}(U) \\ & = \text{vec} \left(E_{\phi\phi_0}^T F E_{rr_0} + \alpha \left(E_{\phi\phi_0}^{\cos^2 \phi} \right)^T \mathbf{h} M_{r_0}^T + \beta \left(E_{\phi\phi_0}^{\sin^2 \phi} \right)^T \mathbf{h} M_{r_0}^T - \frac{\alpha - \beta}{2} M_{\sin 2\phi_0} \mathbf{g}^T E_{rr_0} \right) \end{aligned} \quad (2.12)$$

which can be thought of as a linear system

$$\mathbf{A} \mathbf{x} = \mathbf{b}, \quad (2.13)$$

where $\mathbf{x} = \text{vec}(U)$ is an unknown vector of size MN , \mathbf{b} is a known vector also of size MN , and A is a sparse matrix of order MN . The solution \mathbf{x} is then reshaped as an $M \times N$ matrix U which gives the approximate solution $\tilde{u}(r, \phi) = \Phi^T(\phi) U R(r)$ in (2.5). One can consider solving (2.13) in two ways. One way is to get the minimum norm least squares solution $\mathbf{x} = A^\dagger \mathbf{b}$ where A^\dagger is the standard matrix pseudo-inverse or Moore-Penrose inverse of A . This pseudo-inverse exists and unique for any matrix. The solution provided by $A^\dagger \mathbf{b}$ is a least squares minimum norm solution and is called here the l_2 solution. The other way is to get the minimum l_1 -norm solution by linear programming using l_1 -magic [6]. In the latter case, the problem is formulated as

$$\text{Minimize } |x_1| + \dots + |x_{MN}| \text{ subject to } \mathbf{A} \mathbf{x} = \mathbf{b}.$$

Remark 2.1. *It is important to say a few words on the operational matrices M_R^r and E_{rr_0} in (2.8) and (2.10), respectively. In obtaining these matrices, all terms of degree greater than $n = 3$ have been neglected. For the sake of higher accuracy of the solution, these neglected terms can be represented in terms of the radial polynomials in $R(r)$, see(2.7), by projecting on the space spanned by $R(r)$. Alternatively, a Lagrange interpolation polynomial can be constructed using $R(r)$ to represent each of the neglected higher order terms. The calculated coefficients in the representation of these higher order terms can then be used in the integration operational matrix as explained in connection with (3.7) in the next section.*

In the case of the FOPDE in Example 2.2 below, when the projection of higher order terms is not considered, the solution surface with the l_1 method is found to be acceptable when compared with the actual solution but quite distorted with the l_2 method. It is found that projecting these higher order terms on the space of lower order radial polynomials yields solutions with higher accuracy in both the l_1

and l_2 methods. This method of projection has been used to obtain the results in Example 2.2 that are shown in Figure 1 although we have not displayed the updated matrices considering the projections in our calculations above. We have shown this for the case of a second order PDE in Section 3. In the case of a second order PDE, as discussed in Section 3, this method of projection is found to be crucial in getting a satisfactory solution.

Example 2.2. Let $\alpha = 1$, $\beta = -1$, $\gamma = 1$, and $f = e^{r \cos \phi}(1 + r \cos \phi)$ in (2.2) and (2.3). With this choice one can proceed to solve the following initial value problem:

$$r \cos 2\phi \frac{\partial u}{\partial r} - \sin 2\phi \frac{\partial u}{\partial \phi} + u = e^{r \cos \phi}(1 + r \cos \phi) \quad (2.14)$$

subject to the initial conditions

$$\begin{aligned} u(0, \phi) &= 1, \\ u(r, 0) &= e^r. \end{aligned}$$

It can be checked by direct substitution that $u(r, \phi) = e^{r \cos \phi}$ is a solution to the above initial value problem (2.14). By keeping terms of degree at most three in the expansion by Zernike polynomials, an approximation of the actual solution $u(r, \phi) = e^{r \cos \phi}$ is

$$\begin{aligned} \tilde{u}(r, \phi) &= 1 + r \cos \phi + \frac{1}{2}r^2 \cos^2 \phi + \frac{1}{6}r^3 \cos^3 \phi = 1 + r \cos \phi + \frac{1}{4}r^2 \cos 2\phi + \frac{1}{24}r^3 \cos 3\phi + \frac{1}{8}r^3 \cos \phi \\ &= \Phi^T(\phi)UR(r), \end{aligned}$$

where

$$U = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Expanding the forcing function f in terms of the Zernike polynomials, up to an approximation of order three, gives

$$\begin{aligned} f(r, \phi) &= e^{r \cos \phi}(1 + r \cos \phi) = 1 + 2r \cos \phi + \frac{3}{4}r^2 + \frac{3}{4}r^2 \cos 2\phi + \frac{r^3}{2} \cos \phi + \frac{r^3}{6} \cos 3\phi \\ &= \Phi^T(\phi)FR(r) \end{aligned} \quad (2.15)$$

where

$$F = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The other unknowns in (2.11) are the vectors \mathbf{h} and \mathbf{g} which can be substituted as follows. Let

$$\mathbf{h}^T = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0].$$

Then $h(\phi) = \mathbf{h}^T \Phi(\phi) = 1$ as needed. Let

$$\mathbf{g}^T = [1 \ 1 \ 0 \ \frac{1}{2} \ 0 \ \frac{1}{6}].$$

Then

$$\mathbf{g}^T R(r) = [1 \ 1 \ 0 \ \frac{1}{2} \ 0 \ \frac{1}{6}] \begin{bmatrix} 1 \\ r \\ 2r^2 - 1 \\ r^2 \\ 3r^3 - 2r \\ r^3 \end{bmatrix} = 1 + r + \frac{1}{2}r^2 + \frac{1}{6}r^3$$

which can be thought of as the approximation of $g(r) = e^r$ using terms of degree at most three. These specific vectors are then used in (2.12) to solve for the unknown U . The minimum least squares solution using MATLAB is $\mathbf{x} = \text{psinv}(A)\mathbf{y}$ from which converting vector \mathbf{x} to matrix U we get

$$U = \text{vec2mat}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & -0.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which in terms of Zernike polynomials is

$$\begin{aligned} \tilde{u}(r, \phi) &= 1 + \frac{1}{4}r^2 + 0.2r \cos \phi - 0.4(3r^3 - 2r) \cos \phi + \frac{1}{4}r^2 \cos 2\phi \\ &= 1 + r \cos \phi + \frac{1}{4}r^2(1 + \cos 2\phi) - \frac{6}{5}r^3 \cos \phi. \end{aligned}$$

Alternatively, using an l_1 optimization algorithm based on basis pursuit as explained in [6] the solution matrix for U is

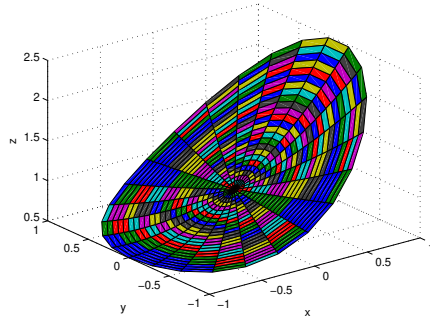
$$U = \text{vec2mat}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which gives

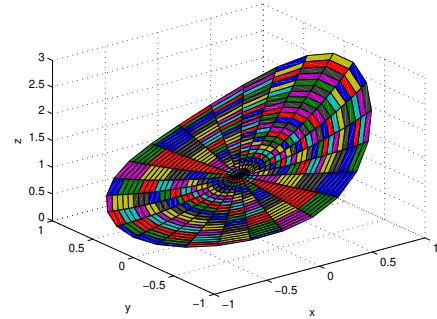
$$\tilde{u}(r, \phi) = 1 + \frac{1}{4}r^2(1 + \cos 2\phi) - \frac{1}{2}(3r^2 - 2r) \cos \phi = 1 + r \cos \phi + \frac{1}{4}r^2(1 + \cos 2\phi) - \frac{3}{2}r^3 \cos \phi.$$

Both of these may be compared with the exact solution mentioned at the start of the example.

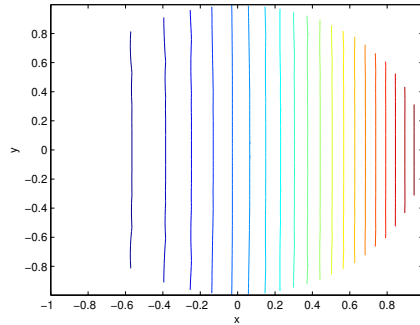
FOPDE surface: 7X6 order I1 approx



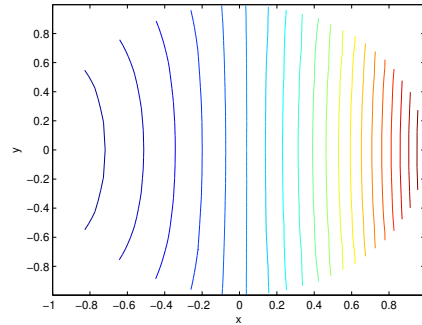
FOPDE surface: 15X20 order I1 approx



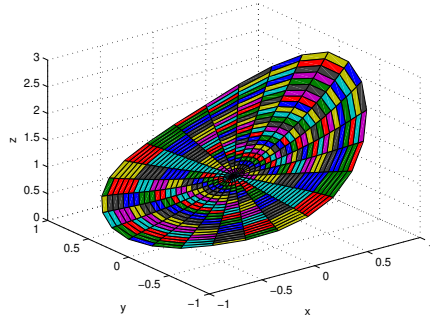
FOPDE contour lines: 7X6 order I1 approx



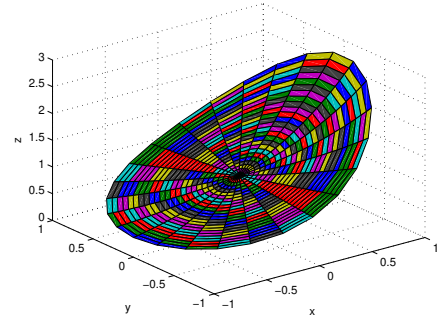
FOPDE contour lines: 15X20 order I1 approx



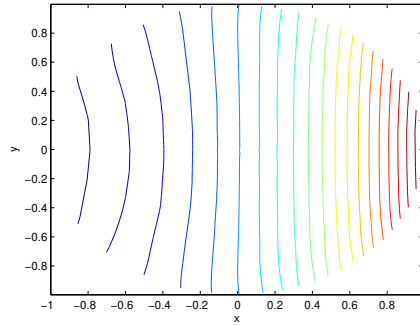
FOPDE surface with 21X36 order I1 approx



FOPDE analytical sol: surface plot



FOPDE contour lines with 21X36 order I1 approx



FOPDE analytical sol: contour lines

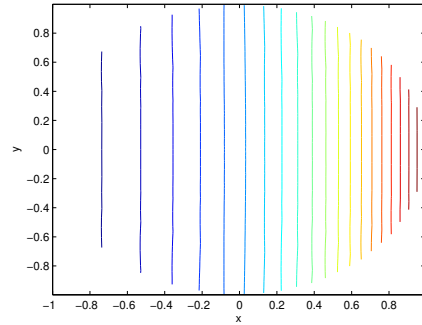


Figure 1: FOPDE: Solution surfaces and contour lines.

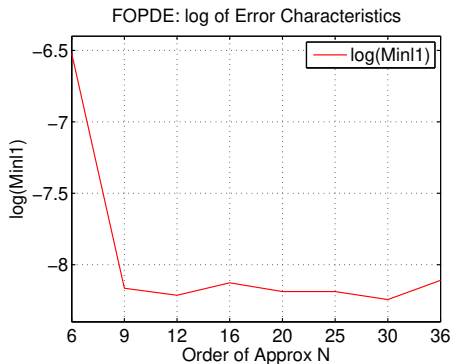


Figure 2: FOPDE solution: log of error curve with Lagrange approximation of higher order terms.

Table 1: Errors in solving FOPDE

Order of Zernike pol.	7 × 6	9 × 9	11 × 12	13 × 16	15 × 20	17 × 25	19 × 30	21 × 36
l_2 -error (order 10^{-4})	783.0700	13.2580	33.1340	24.0200	239.6800	3.1834	187.5100	5.8121
l_1 -error (order 10^{-4})	14.7180	2.8439	2.7084	2.9555	2.7794	2.7793	2.6274	3.0066

Error estimates Recall that the sizes of $\Phi(\phi)$ and $R(r)$ are M and N , respectively. To study the error for different orders of approximation, the solution of FOPDE (2.14) has been determined numerically for the pair (M, N) to be $(7,6)$, $(9,9)$, $(11,12)$, $(13,16)$, $(15,20)$, $(17,25)$, $(19,30)$ and $(21,36)$. The solution surfaces have then been compared with that generated by the exact solution. To do this, one needs to solve (2.13): $A\mathbf{x} = \mathbf{b}$. Recall that A is a sparse matrix of order MN , and $M = 21$ and $N = 36$ for the highest order of approximation considered. The surfaces and contour lines for values of (M, N) equal to $(7,6)$, $(15,20)$ and $(21,36)$ obtained by the minimum l_1 -norm solution are shown in Figure 1 which may be compared with the actual solution surface and contour lines, also displayed in Figure 1. The surfaces provided by the minimum norm least squares solution are much inferior to the minimum l_1 -norm solution, and not shown here. The Mean Square Error (MSE) between the actual and the computed solution is given by the mathematical formula:

$$\text{MSE} = \frac{1}{m_1 n_1} \sum_1^{m_1} \sum_1^{n_1} [X(i, j) - X_c(i, j)]^2,$$

where $X(x, y)$ represents the actual solution surface, $X_c(x, y)$, the computed surface, and $m_1 \times n_1$ are the number of grid points on the surface. By comparing the minimum l_1 -norm solution with that of the minimum norm least squares solution for different orders of approximation it is found that the minimum l_1 -norm solution is much superior as can be inferred from Table 1. The log of the error in the minimum l_1 -norm solution is shown in Figure 2. In Figure 2, the eight distinct points on the error curves correspond to the values of (M, N) given at the start of this paragraph. The minimum l_1 -norm solution has less error when the higher order terms are projected on the space of lower degree polynomials by using Lagrange interpolation, see Remark 2.1.

3 Solving second order partial differential equations

The general form for a linear second order partial differential equation (SOPDE) is

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + a_0 u = f(x, y) \quad (3.1)$$

where a, b, c, a_1, a_2, a_0 , and f are continuous functions of x and y . Since we are motivated by problems involving SOPDEs arising in circular regions such as in a refracted wavefront through an optical system, we shall consider a SOPDE that is invariant under rotations of the coordinate axes about the origin. We have considered below a rotational invariant second order linear PDE with discontinuous BCs, and this special type includes Poisson's PDE appearing in physical systems. In addition to the above mentioned motivation, this choice is also for the purpose of demonstration. However, it is important to emphasize that the proposed method can be applied to PDEs that are not rotational invariant such as parabolic and hyperbolic PDEs, and following the procedure described below one can set up an algebraic equation as in (3.32) that will lead to the solution of the desired PDE. To demonstrate our method, we shall consider

$$\Delta u + \alpha \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^2 u + \beta \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) u + \gamma u = f. \quad (3.2)$$

In polar coordinates r and ϕ , equation (3.2) becomes

$$(1 + \alpha r^2) \frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r} + (\alpha + \beta)r \right) \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \gamma u = f. \quad (3.3)$$

The aim is to solve this SOPDE by integration operational matrix using Zernike polynomials in the region $0 < r < r_0$ and $0 < \phi < 2\pi$ with the given boundary conditions,

$$u(r_0, \phi) = g(\phi), \quad \left. \frac{\partial u(r, \phi)}{\partial r} \right|_{r=r_0} = h(\phi), \quad u(r, \phi_0) = p(r), \quad \left. \frac{\partial u(r, \phi)}{\partial \phi} \right|_{\phi=\phi_0} = q(r),$$

where any of the functions may have discontinuities. Multiplying both sides of (3.3) by r^2 gives

$$r^2(1 + \alpha r^2) \frac{\partial^2 u}{\partial r^2} + (r + \alpha r^3 + \beta r^3) \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} + \gamma r^2 u = r^2 f. \quad (3.4)$$

Integrating twice with respect to r from r_0 to r , the first, third and sixth terms in the left side of (3.4), which do not contain the parameters α, β and γ , one gets

$$\begin{aligned} & \int_{r_0}^r \int_{r_0}^r r^2 \frac{\partial^2 u}{\partial r^2} (dr)^2 + \int_{r_0}^r \int_{r_0}^r r \frac{\partial u}{\partial r} (dr)^2 + \int_{r_0}^r \int_{r_0}^r \frac{\partial^2 u}{\partial \phi^2} (dr)^2 \\ = & r^2 u(r, \phi) - r_0^2 g(\phi) - 3 \int_{r_0}^r r u(r, \phi) dr - \int_{r_0}^r r_0^2 h(\phi) dr \\ & + \int_{r_0}^r r_0 g(\phi) dr + \int_{r_0}^r \int_{r_0}^r u(r, \phi) (dr)^2 + \int_{r_0}^r \int_{r_0}^r \frac{\partial^2 u}{\partial \phi^2} (dr)^2. \end{aligned}$$

Again, integrating the above expression with respect to ϕ twice from ϕ_0 to ϕ gives

$$\begin{aligned}
& \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r^2 u(r, \phi) (d\phi)^2 - \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r_0^2 g(\phi) (d\phi)^2 - 3 \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r u(r, \phi) dr (d\phi)^2 \\
& - \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0^2 h(\phi) dr (d\phi)^2 + \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0 g(\phi) dr (d\phi)^2 \\
& + \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r u(r, \phi) (dr)^2 (d\phi)^2 + \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r \frac{\partial^2 u}{\partial \phi^2} (dr)^2 (d\phi)^2. \tag{3.5}
\end{aligned}$$

Expand the solution $u(r, \phi)$ of (3.2) in terms of Zernike polynomials up to some order (m, n) where $n \geq m$ and $n - m$ is even. This gives an approximation of u as $\tilde{u}(r, \phi) = \Phi^T(\phi) U R(r)$, where $\Phi(\phi)$ is a matrix of size $M \times 1$, $R(r)$ is of size $N \times 1$, $M = 2m + 1$ and N is the number of radial polynomials of degree less than or equal to n . Then each term of (3.5) has the following simplifications. The first term in (3.5) can be written as [§]

$$\int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r^2 u(r, \phi) (d\phi)^2 = \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r^2 (\Phi^T(\phi) U R(r)) (d\phi)^2 = \Phi^T(\phi) E_{D\phi\phi_0}^T U M_R^2 R(r), \tag{3.6}$$

where M_R^2 is the matrix representation of $r^2 R(r)$ with respect to $R(r)$, $E_{D\phi\phi_0}$ is the IOM of double integration of $\Phi(\phi)$ and is $E_{D\phi\phi_0} = E_{\phi\phi_0}^2$, and $E_{D\phi\phi_0}(1, 1) = 2\pi^2/3$. Using radial polynomials up to order (3, 3),

$$r^2 R(r) = \begin{bmatrix} r^2 \\ r^3 \\ 2r^4 - r^2 \\ r^4 \\ 3r^5 - 2r^3 \\ r^5 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ r \\ 2r^2 - 1 \\ r^2 \\ 3r^3 - 2r \\ r^3 \end{bmatrix} = M_R^2 R(r). \tag{3.7}$$

As $R(r)$ contains radial polynomials of maximum degree 3, the terms r^4 and r^5 are ignored in the above. To obtain a better solution, we approximate r^4 and r^5 in terms of the set $\{1, r, r^2, r^3\}$ by Lagrange interpolation formula with equally spaced nodes $(0, 1/3, 2/3, 1)$ in the interval $0 \leq r \leq 1$ as,

$$r^4 \approx 2r^3 - \frac{11}{9}r^2 + \frac{2}{9}r, \quad r^5 \approx \frac{1}{9}(25r^3 - 20r^2 + 4r), \tag{3.8}$$

and incorporate them in the IOM representation whenever they are encountered. Then,

$$2r^4 - r^2 \approx \frac{4}{9}r - \frac{31}{9}r^2 + 4r^3, \quad 3r^5 - 2r^3 \approx \frac{4}{3}r - \frac{26}{3}r^2 + \frac{25}{3}r^3.$$

Consequently, in (3.7), we upgrade M_R^2 to

$$M_R^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{4}{9} & 0 & -\frac{31}{9} & 0 & 4 \\ 0 & \frac{4}{9} & 0 & -\frac{11}{9} & 0 & 2 \\ 0 & \frac{4}{9} & 0 & -\frac{26}{3} & 0 & \frac{25}{3} \\ 0 & \frac{4}{9} & 0 & -\frac{20}{9} & 0 & \frac{25}{9} \end{bmatrix}.$$

[§]Strictly speaking, the resulting matrix representation is an approximation of some integral. However, we will always use the equality sign to express that the operators on the right side of the equality stand for corresponding integrals on the left side.

If these higher order terms are completely ignored (instead of considering their projection), then this approach of solving SOPDE using Zernike polynomials with IOM will fail.

The second term in (3.5) is

$$- \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r_0^2 g(\phi) (d\phi)^2 = - \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \Phi^T(\phi) \mathbf{g} r_0^2 (d\phi)^2 = - \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^2}^T R(r),$$

where $g(\phi) = \Phi^T(\phi) \mathbf{g}$, is the representation of $u(r_0, \phi) = g(\phi)$ in terms of trigonometric functions, and \mathbf{g} is an $M \times 1$ vector. $M_{r_0^2}$ is an $N \times 1$ vector with first element r_0^2 and others zero. Using Zernike polynomials up to degree three:

$$r_0^2 = [r_0^2 \ 0 \ 0 \ 0 \ 0 \ 0] \begin{bmatrix} 1 \\ r \\ 2r^2 - 1 \\ r^2 \\ 3r^3 - 2r \\ r^3 \end{bmatrix} = M_{r_0^2}^T R(r).$$

The third term in (3.5) is

$$- 3 \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r u(r, \phi) dr (d\phi)^2 = - \Phi^T(\phi) E_{D\phi\phi_0}^T U E_{rr_0}^r R(r), \quad (3.9)$$

where $E_{rr_0}^r$ is the IOM of $rR(r)$ in which powers of r higher than n are included after approximating in terms of lower powers of r , by the Lagrange interpolation formula with equally spaced nodes $(0, 1/3, 2/3, 1)$ in the interval $0 \leq r \leq 1$, as mentioned earlier.

The fourth, fifth, and sixth terms in (3.5) are, respectively,

$$- \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r h(\phi) r_0^2 dr (d\phi)^2 = - \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{h} M_{r_0^2}^T E_{rr_0} R(r), \quad (3.10)$$

$$\int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r g(\phi) r_0 dr (d\phi)^2 = \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0}^T E_{rr_0} R(r), \quad (3.11)$$

$$\int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r u(r, \phi) (dr)^2 (d\phi)^2 = \Phi^T(\phi) E_{D\phi\phi_0}^T U E_{Drr_0} R(r). \quad (3.12)$$

The seventh term in (3.5) is

$$\begin{aligned} \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r \frac{\partial^2 u}{\partial \phi^2} (dr)^2 (d\phi)^2 &= \int_{r_0}^r \int_{r_0}^r [u(r, \phi) - u(r, \phi_0)] (dr)^2 - \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r q(r) (dr)^2 d\phi \\ &= \Phi^T(\phi) U E_{Drr_0} R(r) - \Phi^T(\phi) \mathbf{e}_1 \mathbf{p}^T E_{Drr_0} R(r) \\ &\quad - \Phi^T(\phi) E_{\phi\phi_0}^T \mathbf{e}_1 \mathbf{q}^T E_{Drr_0} R(r), \end{aligned} \quad (3.13)$$

where $\mathbf{e}_1 = [1 \ 0 \ \dots \ 0]^T$ is of size $M \times 1$. Inserting the simplifications (3.6)-(3.13) in (3.5), gives

$$\begin{aligned} \Phi^T(\phi) [E_{D\phi\phi_0}^T U M_R^2 - E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^2}^T - 3E_{D\phi\phi_0}^T U E_{rr_0}^r - E_{D\phi\phi_0}^T \mathbf{h} (M_{r_0^2})^T E_{rr_0} + E_{D\phi\phi_0}^T \mathbf{g} (M_{r_0})^T E_{rr_0} \\ + E_{D\phi\phi_0}^T U E_{Drr_0} + I_m U E_{Drr_0} - \mathbf{e}_1 \mathbf{p}^T E_{Drr_0} - E_{\phi\phi_0}^T \mathbf{e}_1 \mathbf{q}^T E_{Drr_0}] R(r). \end{aligned} \quad (3.14)$$

Integrating with respect to r from r_0 to r , the second, fourth, and fifth terms in (3.4), which contain the parameters α and β , one gets

$$\begin{aligned}
& \int_{r_0}^r \left(\alpha r^4 \frac{\partial^2 u(r, \phi)}{\partial r^2} + (\alpha + \beta) r^3 \frac{\partial u(r, \phi)}{\partial r} \right) dr \\
&= \alpha r^4 \frac{\partial u(r, \phi)}{\partial r} - \alpha r_0^4 \frac{\partial u(r, \phi)}{\partial r} \Big|_{r=r_0} + (-3\alpha + \beta) \int_{r_0}^r r^3 \frac{\partial u(r, \phi)}{\partial r} dr \\
&= \alpha r^4 \frac{\partial u(r, \phi)}{\partial r} - \alpha r_0^4 h(\phi) + (-3\alpha + \beta) r^3 u(r, \phi) \\
&\quad - (-3\alpha + \beta) r_0^3 u(r_0, \phi) - 3(-3\alpha + \beta) \int_{r_0}^r r^2 u(r, \phi) dr. \tag{3.15}
\end{aligned}$$

Integrating again with respect to r from r_0 to r gives

$$\int_{r_0}^r \left(\alpha r^4 \frac{\partial u(r, \phi)}{\partial r} + (3\alpha - \beta) r_0^3 g(\phi) - \alpha r^4 h(\phi) - (3\alpha - \beta) r^3 u(r, \phi) + 3(3\alpha - \beta) \int_{r_0}^r r^2 u(r, \phi) \right) dr. \tag{3.16}$$

The first and fourth terms in (3.16) together give

$$\begin{aligned}
& \int_{r_0}^r \left(\alpha r^4 \frac{\partial u(r, \phi)}{\partial r} - (3\alpha - \beta) r^3 u(r, \phi) \right) dr \\
&= \alpha r^4 u(r, \phi) \Big|_{r_0}^r - 4\alpha \int_{r_0}^r r^3 u(r, \phi) dr - (3\alpha - \beta) \int_{r_0}^r r^3 u(r, \phi) dr \\
&= \alpha r^4 u(r, \phi) - \alpha r_0^4 u(r_0, \phi) - (7\alpha - \beta) \int_{r_0}^r r^3 u(r, \phi) dr. \tag{3.17}
\end{aligned}$$

Hence, (3.16) takes the form

$$\begin{aligned}
& \alpha r^4 u(r, \phi) - \alpha r_0^4 g(\phi) - (7\alpha - \beta) \int_{r_0}^r r^3 u(r, \phi) dr + (3\alpha - \beta) r_0^3 \int_{r_0}^r g(\phi) dr \\
& - \alpha \int_{r_0}^r r_0^4 h(\phi) dr + 3(3\alpha - \beta) \int_{r_0}^r \int_{r_0}^r r^2 u(r, \phi) (dr)^2. \tag{3.18}
\end{aligned}$$

Again, integrating the above expression twice with respect to ϕ from ϕ_0 to ϕ ,

$$\begin{aligned}
& \alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r^4 u(r, \phi) (d\phi)^2 - \alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r_0^4 g(\phi) (d\phi)^2 - (7\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r^3 u(r, \phi) dr (d\phi)^2 \\
& + (3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0^3 g(\phi) dr (d\phi)^2 - \alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0^4 h(\phi) dr (d\phi)^2 \\
& + 3(3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r r^2 u(r, \phi) (dr)^2 (d\phi)^2. \tag{3.19}
\end{aligned}$$

Write each term in (3.19) in terms of IOM noting that $u(r, \phi) = \Phi(\phi)^T U R(r)$ and $M_R^{r^4}$ is the matrix representation of $r^4 R(r)$ with respect to $R(r)$. The first term in (3.19) is

$$\alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r^4 u(r, \phi) (d\phi)^2 = \alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \Phi^T(\phi) U M_R^{r^4} R(r) (d\phi)^2 = \alpha \Phi^T(\phi) E_{D\phi\phi_0}^T U M_R^{r^4} R(r). \tag{3.20}$$

The second term in (3.19) is

$$-\alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} r_0^4 g(\phi) (d\phi)^2 = -\alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \Phi^T(\phi) \mathbf{g} M_R^{r_0^4} R(r) (d\phi)^2 = -\alpha \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^4}^T R(r). \quad (3.21)$$

The third term in (3.19) is

$$\begin{aligned} -(7\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r^3 u(r, \phi) dr (d\phi)^2 &= -(7\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \Phi^T(\phi) U(r^3 R(r)) dr (d\phi)^2 \\ &= -(7\alpha - \beta) \Phi^T(\phi) E_{D\phi\phi_0}^T U E_{rr_0}^3 R(r). \end{aligned} \quad (3.22)$$

The fourth term in (3.19) is

$$\begin{aligned} (3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0^3 g(\phi) dr (d\phi)^2 &= (3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \Phi^T(\phi) \mathbf{g} M_{r_0^3} R(r) dr (d\phi)^2 \\ &= (3\alpha - \beta) \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^3} E_{rr_0} R(r). \end{aligned} \quad (3.23)$$

The fifth term in (3.19) is

$$\begin{aligned} -\alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r r_0^4 h(\phi) dr (d\phi)^2 &= -\alpha \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \Phi^T(\phi) \mathbf{h} M_{r_0^4} R(r) dr (d\phi)^2 \\ &= -\alpha \Phi^T(\phi) E_{D\phi\phi_0}^T \mathbf{h} M_{r_0^4} E_{rr_0} R(r). \end{aligned} \quad (3.24)$$

The sixth term in (3.19) is

$$\begin{aligned} 3(3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r r^2 u(r, \phi) (dr)^2 (d\phi)^2 &= 3(3\alpha - \beta) \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r \Phi^T(\phi) U r^2 R(r) (dr)^2 (d\phi)^2 \\ &= 3(3\alpha - \beta) \Phi^T(\phi) E_{D\phi\phi_0}^T U E_{Drr_0}^2 R(r). \end{aligned} \quad (3.25)$$

Combining the terms (3.20)-(3.25), the term (3.19) takes the form, in terms of IOMs,

$$\begin{aligned} \Phi^T(\phi) [\alpha E_{D\phi\phi_0}^T U M_R^{r_0^4} - \alpha E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^4}^T - (7\alpha - \beta) E_{D\phi\phi_0}^T U E_{rr_0}^3 + (3\alpha - \beta) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^3} E_{rr_0} \\ - \alpha E_{D\phi\phi_0}^T \mathbf{h} M_{r_0^4} E_{rr_0} + (3\alpha - \beta) (\phi) E_{D\phi\phi_0}^T U E_{Drr_0}^2] R(r). \end{aligned} \quad (3.26)$$

The last two terms in (3.4) in terms of integration operational matrices after integrating twice with respect to ϕ from ϕ_0 to ϕ and again twice with respect to r from r_0 to r , are

$$\begin{aligned} \gamma \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r r^2 u(r, \phi) (dr)^2 (d\phi)^2 &= \gamma \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r \Phi^T(\phi) U(r^2 R(r)) (dr)^2 (d\phi)^2 \\ &= \gamma \Phi^T(\phi) E_{D\phi\phi_0}^T U E_{Drr_0}^2 R(r), \end{aligned} \quad (3.27)$$

$$\begin{aligned} \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r r^2 f(r, \phi) (dr)^2 (d\phi)^2 &= \int_{\phi_0}^{\phi} \int_{\phi_0}^{\phi} \int_{r_0}^r \int_{r_0}^r \Phi^T(\phi) F(r^2 R(r)) (dr)^2 (d\phi)^2 \\ &= \Phi^T(\phi) E_{D\phi\phi_0}^T F E_{Drr_0}^2 R(r). \end{aligned} \quad (3.28)$$

Combining the terms (3.14), (3.26), (3.27), and (3.28), the equation (3.3) finally takes the form

$$\begin{aligned}
& \Phi^T(\phi)[E_{D\phi\phi_0}^T U M_R^{r^2} - E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^2}^T - 3E_{D\phi\phi_0}^T U E_{rr_0}^r - E_{D\phi\phi_0}^T \mathbf{h}(M_{r_0^2})^T E_{rr_0} \\
& + E_{D\phi\phi_0}^T \mathbf{g}(M_{r_0})^T E_{rr_0} + E_{D\phi\phi_0}^T U E_{Drr_0} + I_m U E_{Drr_0} - \mathbf{e}_1 \mathbf{p}^T E_{Drr_0} - E_{\phi\phi_0}^T \mathbf{e}_1 \mathbf{q}^T E_{Drr_0} + \alpha E_{D\phi\phi_0}^T U M_R^{r^4} \\
& - \alpha E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^4}^T - (7\alpha - \beta) E_{D\phi\phi_0}^T U E_{rr_0}^{r^3} + (3\alpha - \beta) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^3} E_{rr_0} - \alpha E_{D\phi\phi_0}^T \mathbf{h} M_{r_0^4} E_{rr_0} \\
& + (3\alpha - \beta) E_{D\phi\phi_0}^T U E_{Drr_0}^{r^2} + \gamma E_{D\phi\phi_0}^T U E_{Drr_0}^{r^2}] R(r) = \Phi^T(\phi)[E_{D\phi\phi_0}^T F E_{Drr_0}^{r^2}] R(r). \tag{3.29}
\end{aligned}$$

Based on (3.29), define matrices A and Y as,

$$\begin{aligned}
A = & (M_R^{r^2})^T \otimes E_{D\phi\phi_0}^T - 3(E_{rr_0}^r)^T \otimes E_{D\phi\phi_0}^T + (E_{Drr_0})^T \otimes E_{D\phi\phi_0}^T + (E_{Drr_0})^T \otimes I_m + \alpha (M_R^{r^4})^T \otimes E_{D\phi\phi_0}^T \\
& - (7\alpha - \beta) (E_{rr_0}^{r^3})^T \otimes E_{D\phi\phi_0}^T + (3\alpha - \beta) (E_{Drr_0}^{r^2})^T \otimes E_{D\phi\phi_0}^T + \gamma (E_{Drr_0}^{r^2})^T E_{D\phi\phi_0}^T, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
Y = & E_{D\phi\phi_0}^T F E_{Drr_0}^{r^2} + E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^2}^T + E_{D\phi\phi_0}^T \mathbf{h}(M_{r_0^2})^T E_{rr_0} - E_{D\phi\phi_0}^T \mathbf{g}(M_{r_0})^T E_{rr_0} + \mathbf{e}_1 \mathbf{p}^T E_{Drr_0} \\
& + E_{\phi\phi_0}^T \mathbf{e}_1 \mathbf{q}^T E_{Drr_0} + \alpha E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^4}^T - (3\alpha - \beta) E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^3} E_{rr_0} + \alpha E_{D\phi\phi_0}^T \mathbf{h} M_{r_0^4} E_{rr_0}. \tag{3.31}
\end{aligned}$$

For Y given in (3.31), define the vector $\mathbf{b} = \text{vec}(Y)$. Let $\mathbf{x} = \text{vec}(U)$ be the vector that is unknown. Using the matrix A in (3.30), (3.29) can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{3.32}$$

Therefore, the solution of the PDE in (3.2) can be found by solving a system of linear equations given by (3.32) in which A is a sparse matrix of order MN . The solution \mathbf{x} , an $MN \times 1$ matrix, is then reshaped as an $M \times N$ matrix U which gives $\tilde{u}(r, \phi) = \Phi^T(\phi)UR(r)$, an approximate solution of (3.2).

For the Laplace equation, $\alpha = \beta = \gamma = 0$, and also $F = 0$. In this case (3.30) and (3.31) simplify to, respectively,

$$A = (M_R^{r^2})^T \otimes E_{D\phi\phi_0}^T - 3(E_{rr_0}^r)^T \otimes E_{D\phi\phi_0}^T + (E_{Drr_0})^T \otimes E_{D\phi\phi_0}^T + (E_{Drr_0})^T \otimes I_m, \tag{3.33}$$

$$Y = E_{D\phi\phi_0}^T \mathbf{g} M_{r_0^2}^T + E_{D\phi\phi_0}^T \mathbf{h}(M_{r_0^2})^T E_{rr_0} - E_{D\phi\phi_0}^T \mathbf{g}(M_{r_0})^T E_{rr_0} + \mathbf{e}_1 \mathbf{p}^T E_{Drr_0} + E_{\phi\phi_0}^T \mathbf{e}_1 \mathbf{q}^T E_{Drr_0}. \tag{3.34}$$

To demonstrate the accuracy of this method, in Example 3.1 below, the corresponding linear equation obtained in (3.32) is solved in two ways. In one the standard matrix pseudo-inverse of A is used, in which case $\mathbf{x} = A^\dagger \mathbf{b}$, and in the other an l_1 -optimization algorithm is used. Both have been implemented using Matlab.

Example 3.1 (Numerical Solution of a Second Order PDE). Consider the second order PDE,

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = 0, \tag{3.35}$$

Table 2: Density of the sparse matrix A

Order of Zernike pol. $M \times N$	Order of A $MN \times MN$	Non-zero elements in A x	Sparsity of A x/M^2N^2
7×6	42×42	502	0.2846
9×9	81×81	1275	0.1943
11×12	132×132	2680	0.1538
13×16	208×208	5102	0.1179
15×20	300×300	8745	0.0972
17×25	425×425	14306	0.0792
19×30	570×570	22009	0.0677
21×36	756×756	33048	0.0578

in a circular region of radius a subject to the boundary conditions (BCs)

$$\begin{aligned}
g(\phi) &:= u(r_0, \phi) = \begin{cases} V_0, & 0 < \phi < \pi; \\ 0, & \pi < \phi < 2\pi; \end{cases} \\
h(\phi) &= \left. \frac{\partial u(r, \phi)}{\partial r} \right|_{r=a} = \frac{V_0}{\pi a \sin \phi}; \\
p(r) &= u(r, \phi_0) = V_0 \left\{ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{2ar \sin \phi_0}{(a^2 - r^2)} \right\}; \\
q(r) &= \left. \frac{\partial u(r, \phi)}{\partial \phi} \right|_{\phi=\phi_0} = \frac{V_0}{\pi} \frac{2ar \cos \phi_0 (a^2 - r^2)}{(a^2 - r^2)^2 + 4a^2 r^2 \sin^2 \phi_0}.
\end{aligned}$$

This is a Laplace equation in polar coordinates and appears in determining the potential distribution in a horizontal cylindrical region with axial symmetry when the upper half is maintained at a potential V_0 and the lower half at zero potential. Without any loss of generality, assume that $V_0 = 1$, $r_0 = a = 1$, and $\phi_0 = 0$. So, the BCs in terms of the Zernike polynomials are,

$$\begin{aligned}
g(\phi) &= u(r_0, \phi) = \frac{1}{2} + \frac{2}{\pi} \sum_{i=0}^{\infty} \frac{\sin(2i+1)\phi}{(2i+1)} = \mathbf{g}^T \Phi(\phi), \\
h(\phi) &= \left. \frac{\partial u(r, \phi)}{\partial r} \right|_{r=r_0} = \frac{2}{\pi} \sum_{i=0}^{\infty} \sin(2i+1)\phi = \mathbf{h}^T \Phi(\phi), \\
p(r) &= u(r, \phi_0) = \frac{1}{2} = \mathbf{p}^T R(r), \\
q(r) &= \left. \frac{\partial u(r, \phi)}{\partial \phi} \right|_{\phi=\phi_0} = \frac{2r}{\pi(1-r^2)} = \mathbf{q}^T R(r),
\end{aligned}$$

where $\mathbf{p} = [\frac{1}{2} \ 0 \ 0 \ 0 \ 0 \ 0 \ \dots]^T$, $\mathbf{q} = \frac{2}{\pi}[0 \ 1 \ 0 \ 0 \ 0 \ 1 \ \dots]^T$, $\mathbf{g} = \frac{2}{\pi}[\pi/4 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1/3 \ \dots]^T$, and $\mathbf{h} = \frac{2}{\pi}[0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ \dots]^T$. With the above BCs, the solution of the Laplace equation (3.35) is obtained numerically using (3.33) and (3.34) for values (7,6), (9,9), (11,12), (13,16), (15,20), (17,25), (19,30) and (21,36) of the pair (M, N) . The Zernike polynomial based solution is compared with the exact solution

Table 3: Errors in solving SOPDE

Order of Zernike pol.	7 × 6	9 × 9	11 × 12	13 × 16	15 × 20	17 × 25	19 × 30	21 × 36
ℓ_2 -error (order 10^{-4})	17.1369	15.9970	11.6488	7.3285	4.5292	7.4972	5.0259	8.3604
ℓ_1 -error (order 10^{-4})	14.9311	14.9311	7.3285	7.3285	4.4431	4.4431	3.0503	3.0499

and the errors are listed in Table 3 for l_1 as well as l_2 methods, the l_1 -errors being much lower. It appears from Figure 5 that the error decreases exponentially with the increase of ZRP degrees. The solution surfaces and their contour lines for only three values (7,6), (15,20), and (21,36) of (M, N) are shown in Figures 3-4 for the Zernike polynomial based solutions, along with the solution by the product method, and the exact Poisson integral solution from where one can compare the different solutions. The eight distinct points on the error curve in Figure 5 correspond to the values of (M, N) as mentioned above. It appears that among the approximate solutions, the l_1 method is better, and the accuracy improves with higher order terms. The proposed method compares favorably with the exact solution particularly if $(M, N) = (21, 36)$. To justify the use of an l_1 -algorithm for sparse solutions, the sparsity, i.e. the density of the sparse matrix A defined as the number of non-zero elements in A is computed, and these densities corresponding to the order of the Zernike polynomials used are shown in Table 2. We reiterate that if the higher order terms are not approximated by projecting on the space spanned by the lower order terms, the solution in this second order case is nowhere near the true solution.

Computational complexity Recall that n denotes the maximum degree of radial polynomials used. To compute the Zernike coefficients of the forcing function f , the cost of computation is $O(n^3)$, see [4]. In (2.12) and (3.29), the number of operations mainly comes from calculating the tensor product which is $O(n^4)$. However, it is important to note that the matrices obtained from the tensor products in (2.12) or after (3.29) have to be computed just once and thereafter the same matrices can be used to solve different problems with other boundary conditions. The linear system $A\mathbf{x} = \mathbf{b}$, where A is a sparse matrix, can be efficiently solved using an l_1 -minimization algorithm.

4 Rate of decay of Zernike coefficients

Recall from Section 1 that the coefficients in a Zernike expansion as given in (1.6) are

$$A_{nm} = \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi) \cos m\phi R_n^m(r) r d\phi dr,$$

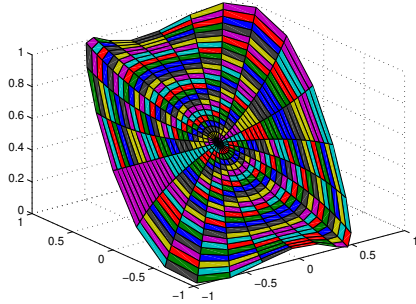
$$B_{nm} = \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi) \sin m\phi R_n^m(r) r d\phi dr.$$

The rate of decay of the A_{mns} and B_{mns} will be calculated here for certain types of functions. These are given in Theorem 4.1 and Theorem 4.3 below.

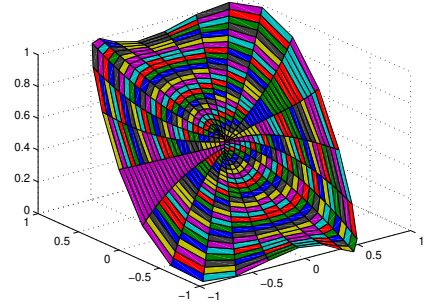
Define the inner product on the unit disk as

$$\langle f, g \rangle = \int_0^1 \int_0^{2\pi} f(r, \phi) \bar{g}(r, \phi) r dr d\phi.$$

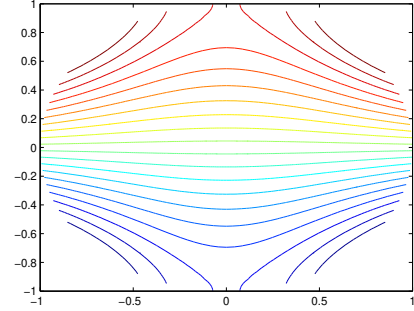
SOPDE surface with 7×6 order I1 approx.



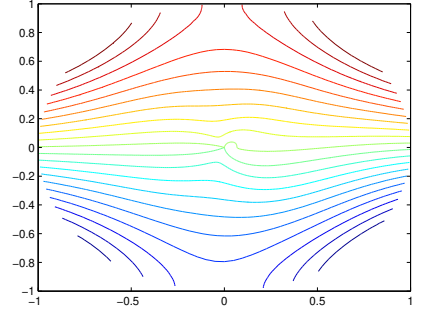
SOPDE surface with 7×6 order I2 approx.



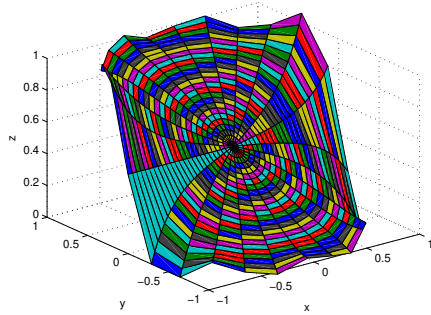
SOPDE Contour lines with 7×6 order I1 approx.



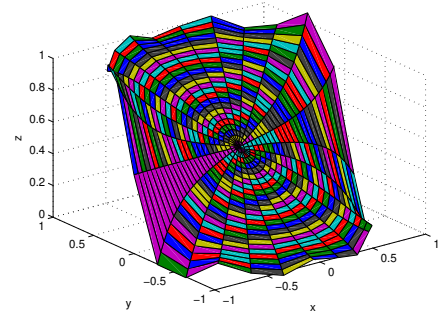
SOPDE Contour lines with 7×6 order I2 approx.



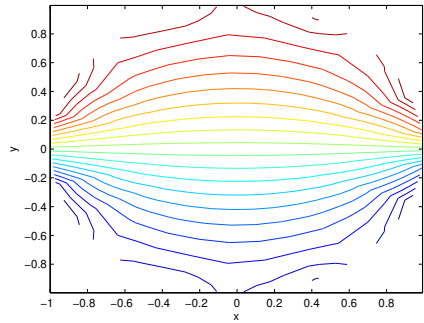
SOPDE surface with 15×20 order I1 approx.



SOPDE surface with 15×20 order I2 approx.



SOPDE Contour lines with 15×20 order I1 approx.



SOPDE Contour lines with 15×20 order I2 approx.

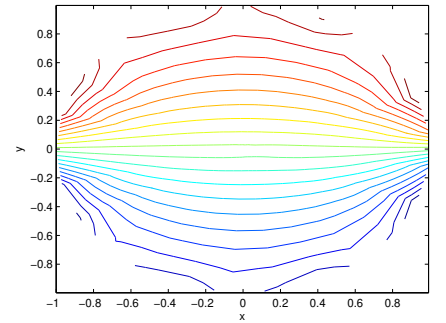
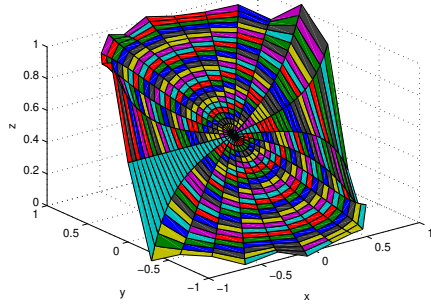
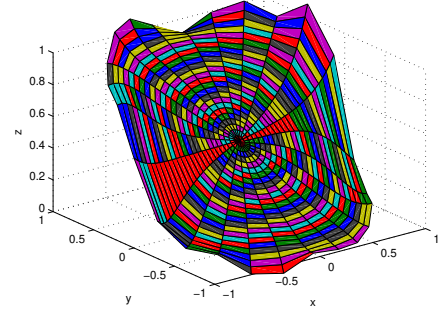


Figure 3: SOPDE Solution: Solution surface and contour lines.

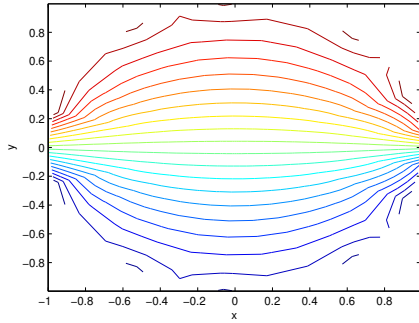
SOPDE surface with 21×36 order I1 approx.



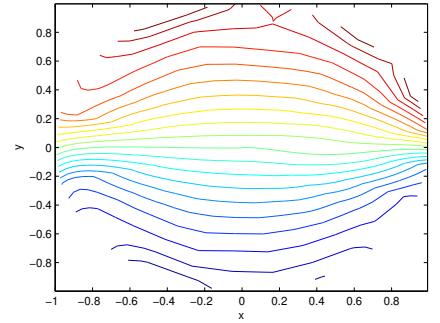
SOPDE surface with 21×36 order I2 approx.



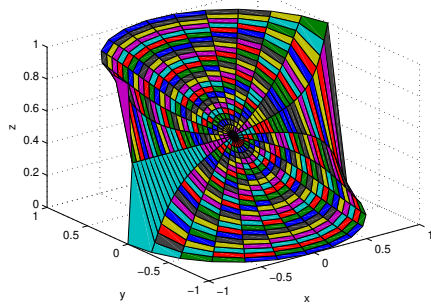
SOPDE Contour lines with 21×36 order I1 approx.



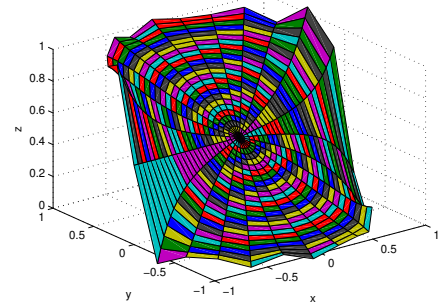
SOPDE Contour lines with 21×36 order I2 approx.



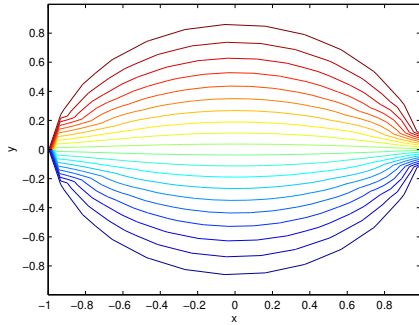
SOPDE analytical sol: surface plot



Surface: product method upto 10φ terms



SOPDE analytical sol: contour lines



Contour lines: product method upto 10φ terms

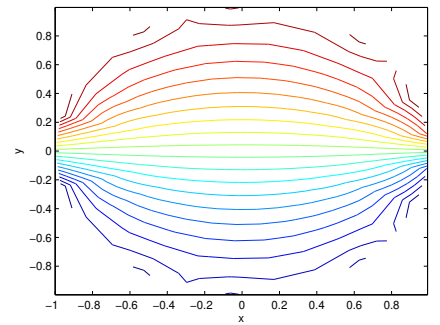


Figure 4: SOPDE Solution: Solution surface and contour lines.

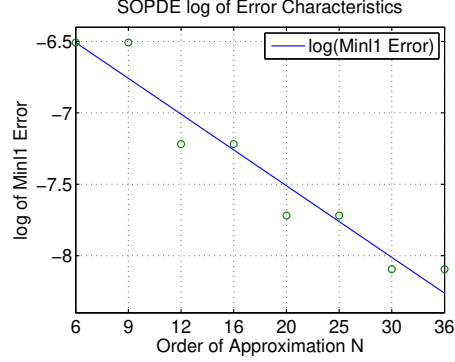


Figure 5: SOPDE Solution: log of error curve.

The associated norm is

$$\|f\| = \sqrt{\int_0^1 \int_0^{2\pi} |f(r, \phi)|^2 r dr d\phi}.$$

Define

$$\left\{ \begin{array}{c} {}^o U_n^m \\ {}^e U_n^m \end{array} \right\} := \left\{ R_n^m(r) \begin{array}{c} \sin m\phi \\ \cos m\phi \end{array} \right\}.$$

As given in [20] and [3]

$$\int_0^1 R_n^m(r) R_{n'}^m(r) r dr = \frac{\delta_{nn'}}{2n+2}$$

which means

$$\|R_n^m\|^2 = \frac{1}{2n+2}.$$

As discussed in [20, 3], the set $\left\{ \begin{array}{c} {}^o U_n^m \\ {}^e U_n^m \end{array} \right\}$ forms an orthogonal basis for the space of square integrable functions in the unit disk. Since $\int_0^{2\pi} d\phi = 2\pi$, $\int_0^{2\pi} \sin^2 m\phi d\phi = \pi$, and $\int_0^{2\pi} \cos^2 m\phi d\phi = \pi$, one can normalize to get the following orthonormal basis for the space of square integrable functions in the unit disk

$$\mathcal{B} := \left\{ \frac{\sqrt{2n+2}}{\sqrt{\pi}} R_n^m(r) \begin{array}{c} \sin m\phi \\ \cos m\phi \end{array} \right\}_{\substack{n=0,1,2,\dots \\ 1 \leq m \leq n \\ n-m \text{ even}}} \cup \left\{ \frac{\sqrt{2n+2}}{\sqrt{2\pi}} R_n^0(r) \right\}_{n=0,2,4,\dots}$$

which can be rewritten more compactly as

$$\mathcal{B} := \frac{\sqrt{\epsilon_m(n+1)}}{\sqrt{\pi}} R_n^m(r) \left\{ \begin{array}{c} \sin m\phi \\ \cos m\phi \end{array} \right\} =: \left\{ \begin{array}{c} {}^o \underline{U}_n^m \\ {}^e \underline{U}_n^m \end{array} \right\}_{\substack{n=0,1,2,\dots \\ 0 \leq m \leq n \\ n-m \text{ even}}}.$$

Any $f(r, \phi)$ defined on the unit disk can be expanded as

$$f(r, \phi) = \sum_{n=0}^{\infty} \sum_{\substack{0 \leq m \leq n \\ n-m \text{ even}}} [\langle f(r, \phi), {}^e \underline{U}_n^m \rangle {}^e \underline{U}_n^m + \langle f(r, \phi), {}^o \underline{U}_n^m \rangle {}^o \underline{U}_n^m]. \quad (4.1)$$

On comparing (4.1) with (1.6) and (1.7), note that

$$A_{nm} = \langle f(r, \phi), {}^e \underline{U}_n^m \rangle, \quad B_{nm} = \langle f(r, \phi), {}^o \underline{U}_n^m \rangle$$

are the coefficients of f with respect to the Zernike polynomial basis. They will be referred to as the Zernike coefficients of f . Define the N th partial sum as

$$S_N f(r, \phi) := \sum_{n=0}^N \sum_{\substack{0 \leq m \leq n \\ n-m \text{ even}}} [(f(r, \phi), {}^e \underline{U}_n^m) {}^e \underline{U}_n^m + (f(r, \phi), {}^o \underline{U}_n^m) {}^o \underline{U}_n^m].$$

Let P_N be the space spanned by Zernike polynomials of radial degree at most N , i.e.,

$$P_N = \left\{ p(r, \phi) : p(r, \phi) = \sum_{n=0}^N \sum_{\substack{0 \leq m \leq n \\ n-m \text{ even}}} [c_{mn} {}^e \underline{U}_n^m + {}^o c_{mn} {}^o \underline{U}_n^m] \right\}.$$

Then $S_N f$ is a polynomial in P_N . Since \mathcal{B} forms an orthonormal basis, by property of orthonormal sets, $S_N f$ is the polynomial of best approximation to f among all polynomials in P_N which means that for any polynomial $p \in P_N$ one has

$$\|f - p\| \geq \|f - S_N f\|$$

with equality holding if and only if $p = S_N f$. By virtue of \mathcal{B} being an orthonormal basis, Parseval's Identity also holds

$$\|f\|^2 = \sum_{n=0}^{\infty} \sum_{\substack{0 \leq m \leq n \\ n-m \text{ even}}} (|\langle f(r, \phi), {}^e \underline{U}_n^m \rangle|^2 + |\langle f(r, \phi), {}^o \underline{U}_n^m \rangle|^2)$$

which in turn gives the Riemann-Lebesgue Lemma in this case:

$$\lim_{m, n \rightarrow \infty} |\langle f(r, \phi), {}^e \underline{U}_n^m \rangle| = \lim_{m, n \rightarrow \infty} |\langle f(r, \phi), {}^o \underline{U}_n^m \rangle| = 0. \quad (4.2)$$

In certain cases, the rate of decay of the coefficients in (4.2) can be obtained as follows.

Denote by $\mathcal{C}^k(B(0, 1))$ the space of functions defined on the unit disk $B(0, 1)$ whose k th order partial derivatives all exist and are continuous on $B(0, 1)$. For convenience, we shall sometimes write \mathcal{C}^k . Suppose that $u(r, \phi) \in \mathcal{C}^2$. Then

$$\begin{aligned} C_{nm} &= \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} u(r, \phi) e^{im\phi} R_n^m(r) r \, d\phi dr \\ &= \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \left[\frac{u(r, \phi) e^{im\phi}}{im} \Big|_0^{2\pi} - \frac{1}{im} \int_0^{2\pi} \frac{\partial u(r, \phi)}{\partial \phi} e^{im\phi} d\phi \right] R_n^m(r) r \, dr \\ &= -\frac{1}{im} \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} \frac{\partial u(r, \phi)}{\partial \phi} e^{im\phi} R_n^m(r) r \, d\phi dr \quad (\text{since } u(r, 2\pi) = u(r, 0)) \\ &= -\frac{1}{im} \frac{\epsilon_m(n+1)}{\pi} \left(-\frac{1}{im} \right) \int_0^1 \int_0^{2\pi} \frac{\partial^2 u(r, \phi)}{\partial \phi^2} e^{im\phi} R_n^m(r) r \, d\phi dr \end{aligned} \quad (4.3)$$

where in the last step we have used the fact that $\frac{\partial u}{\partial \phi}(r, 0) = \frac{\partial u}{\partial \phi}(r, 2\pi)$. The integral on the right side of (4.3) will yield the Zernike coefficients of $\frac{\partial^2 u(r, \phi)}{\partial \phi^2}$. Denoting this by C''_{mn} , one can write

$$C_{mn} = \frac{\epsilon_m(n+1)}{\pi} \left(-\frac{1}{im}\right)^2 C''_{mn}$$

which decays at the rate of m^{-2} . Note that m and n tend to infinity at the same rate. In general, one has the following Theorem 4.1.

Theorem 4.1. *Let $u(r, \phi) \in C^k$. Then the Zernike coefficients A_{nm}, B_{nm} of u decay at the rate of m^{-k} .*

Definition 4.2. *A function $u : U \rightarrow \mathbb{R}$ is said to be Hölder continuous of order λ if for all $x, y \in U$*

$$|u(x) - u(y)| \leq C\|x - y\|^\lambda$$

for some constants λ and C , where $\|\cdot\|$ is the metric on U .

Theorem 4.3. *Let $u(r, \phi)$ be Hölder continuous of order $\lambda \geq 1$. Then the Zernike coefficients A_{nm}, B_{nm} of u decay at least like $m^{-\lambda+1}$.*

Proof. Consider

$$C_{nm} := \frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} u(r, \phi) e^{im\phi} R_n^m(r) r \, d\phi \, dr. \quad (4.4)$$

Rewriting (4.4) gives

$$\begin{aligned} C_{nm} &= -\frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} u(r, \phi) e^{im\phi} e^{i\pi} R_n^m(r) r \, d\phi \, dr \\ &= -\frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} u(r, \phi) e^{im(\phi+\pi/m)} R_n^m(r) r \, d\phi \, dr \\ &= -\frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_{\pi/m}^{2\pi+\pi/m} u(r, \alpha - \pi/m) e^{im\alpha} R_n^m(r) r \, d\alpha \, dr \\ &= -\frac{\epsilon_m(n+1)}{\pi} \int_0^1 \int_0^{2\pi} u(r, \alpha - \pi/m) e^{im\alpha} R_n^m(r) r \, d\alpha \, dr. \end{aligned} \quad (4.5)$$

Adding (4.4) and (4.5) gives

$$\begin{aligned} C_{nm} &= \frac{\epsilon_m(n+1)}{2\pi} \int_0^1 \int_0^{2\pi} [u(r, \phi) - u(r, \phi - \pi/m)] e^{im\phi} R_n^m(r) r \, d\phi \, dr \\ \text{or, } |C_{nm}| &\leq \frac{\epsilon_m(n+1)}{2\pi} \int_0^1 \int_0^{2\pi} |u(r, \phi) - u(r, \phi - \pi/m)| |R_n^m(r) r| \, d\phi \, dr \\ &\leq C \frac{\epsilon_m(n+1)}{2\pi} \left(\frac{\pi}{m}\right)^\lambda \int_0^1 \int_0^{2\pi} |R_n^m(r) r| \, d\phi \, dr \\ &= C \epsilon_m(n+1) \left(\frac{\pi}{m}\right)^\lambda \int_0^1 |R_n^m(r) r| \, dr \\ &\leq C \epsilon_m(n+1) \left(\frac{\pi}{m}\right)^\lambda, \end{aligned}$$

where the last inequality follows from the fact that $|R_n^m(r)r| \leq 1$, since $|R_n^m(r)| \leq 1$ and $0 \leq r \leq 1$, see [22]. Looking at the real and imaginary parts of C_{nm} gives the desired result. Note that m and n tend to infinity at the same rate. \square

5 Appendix

To derive the IOM for the radial parts of Zernike polynomials, the recurrence relation stated in (2.9) will be used. For convenience, this is again provided below.

$$\int_{r_0}^r [R_n^m(r) + R_n^{m+2}(r)] dr = \frac{1}{n+1} [R_{n+1}^{m+1}(r) - R_{n-1}^{m+1}(r)] \Big|_{r_0}^r. \quad (5.1)$$

If all radial polynomials up to degree n are used then the basis vector for the radial parts of Zernike polynomials is

$$R(r) = [R_0^0(r), R_1^1(r), R_2^0(r), R_2^2(r), R_3^1(r), R_3^3(r), \dots, R_n^m(r)], \quad n \in \mathbb{N} \cup \{0\}, \quad 0 \leq n-m, \quad n-m \text{ even}. \quad (5.2)$$

Let i be the degree of a radial polynomial, and let p_i denote the total number of polynomials of degree i . Due to the special structure of the radial polynomials as described in (1.4) of Section 1, the value of p_i is determined as follows. For a given non-negative integer i , the value of p_i is the number of integers j for which $i-j$ is even and non negative. For $i = 0, 1, \dots, n$, let Δ_{i+1} be the $p_i \times p_i$ matrix with ones along the diagonal and also above the main diagonal, and zeros elsewhere. That is,

$$\Delta_{i+1} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{p_i \times p_i}$$

whose inverse is

$$\Delta_{i+1}^{-1} = \begin{bmatrix} 1 & -1 & \cdots & (-1)^{p_i-1} \\ 0 & 1 & -1 & \cdots & (-1)^{p_i-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{p_i \times p_i}$$

Let E_{r1} be a block diagonal matrix of the form

$$E_{r1} = \begin{bmatrix} \Delta_1 & & & \\ & \Delta_2 & & \\ & & \ddots & \\ & & & \Delta_{n+1} \end{bmatrix}.$$

Denote $\int_{r_0}^r R(r) dr$ by E_{rr_0} . Then from (5.1) one can write

$$E_{r1} E_{rr_0} = E_{r2}.$$

The structure of the matrix E_{r1} has been described above. The matrix E_{r2} is a block matrix of order $n+1$, n being the degree of $R_n^m(r)$. Each block in E_{r2} is a submatrix Γ_{kl} and can be represented as

$$E_{r2} = [\Gamma_{kl}]$$

where $\Gamma_{i+1,j+1}$ is of size $p_i \times p_j$. For example, let $n = 5$. Then E_{r_2} is a 6×6 block matrix where all the blocks are null matrices except for the following:

$$\begin{aligned} \Gamma_{11} &= [-R_1^1(r_0)], \Gamma_{12} = [1], \Gamma_{21} = [-\frac{1}{2}R_2^2(r_0)], \Gamma_{23} = [0 \quad \frac{1}{2}], \Gamma_{31} = \begin{bmatrix} -\frac{1}{3}[R_3^1(r_0) - R_1^1(r_0)] \\ -\frac{1}{3}R_3^3(r_0) \end{bmatrix}, \Gamma_{32} = \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}, \\ \Gamma_{34} &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \Gamma_{41} = \begin{bmatrix} -\frac{1}{4}[R_4^2(r_0) - R_2^2(r_0)] \\ -\frac{1}{4}R_4^4(r_0) \end{bmatrix}, \Gamma_{43} = \begin{bmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix}, \Gamma_{45} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \\ \Gamma_{51} &= \begin{bmatrix} -\frac{1}{5}[R_5^1(r_0) - R_3^1(r_0)] \\ -\frac{1}{5}[R_5^3(r_0) - R_3^3(r_0)] \\ -\frac{1}{5}R_5^5(r_0) \end{bmatrix}, \Gamma_{54} = \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix}, \Gamma_{56} = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \\ \Gamma_{61} &= \begin{bmatrix} -\frac{1}{6}[R_6^2(r_0) - R_4^2(r_0)] \\ -\frac{1}{6}[R_6^4(r_0) - R_4^4(r_0)] \\ -\frac{1}{6}R_6^6(r_0) \end{bmatrix}, \Gamma_{65} = \begin{bmatrix} 0 & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

For $0 \leq m \leq n$, if one wants to form the integration operational matrix of $\int_{r_0}^r R_n^m(r) dr$ from (5.1) using only radial polynomials up to degree n , then usually the terms of higher degree are neglected. When n is odd, then for the integrals

$$\int_{r_0}^r R_n^n(r) dr, \int_{r_0}^r R_n^{n-2}(r) dr, \dots, \int_{r_0}^r R_n^1(r) dr,$$

the terms

$$\frac{1}{n+1}R_{n+1}^{n+1}(r), \frac{1}{n+1}[R_{n+1}^{n-1}(r) - R_{n+1}^{n+1}(r)], \dots, \frac{1}{n+1}[R_{n+1}^2(r) - R_{n+1}^4(r) + \dots + (-1)^{\frac{n-1}{2}}R_{n+1}^{n+1}]$$

are neglected, respectively. On the other hand, when n is even, then for the integrals

$$\int_{r_0}^r R_n^n(r) dr, \int_{r_0}^r R_n^{n-2}(r) dr, \dots, \int_{r_0}^r R_n^0(r) dr,$$

the terms,

$$\frac{1}{n+1}R_{n+1}^{n+1}(r), \frac{1}{n+1}[R_{n+1}^{n-1}(r) - R_{n+1}^{n+1}(r)], \dots, \frac{1}{n+1}[R_{n+1}^0(r) - R_{n+1}^2(r) + \dots + (-1)^{\frac{n}{2}}R_{n+1}^{n+1}]$$

are neglected, respectively. In compact form, when one wishes to use only radial polynomials up to degree n to evaluate

$$\int_{r_0}^r R_n^{n-2i}(r) dr,$$

then one neglects

$$\frac{1}{n+1} [R_{n+1}^{n-2i+1} - R_{n+1}^{n-2i+3} + \dots + (-1)^i R_{n+1}^{n+1}(r)], d$$

where

$$i = \begin{cases} 0, 1, \dots, \frac{n-1}{2} & \text{when } n \text{ is odd,} \\ 0, 1, \dots, \frac{n}{2} & \text{when } n \text{ is even.} \end{cases}$$

For better accuracy of results, these neglected terms of degree greater than n can then be represented in terms of the radial polynomials appearing in $R(r)$ of (5.2) as mentioned in Remark 2.1.

6 Conclusion

It is established in this paper that numerical solutions of partial differential equations in circular regions can be successfully done using Zernike polynomials and IOMs which can otherwise be challenging using other orthogonal polynomials. In comparison, using multidimensional block pulse functions and OSOMRI (one shot operational matrix for repeated integrations), with extensive computations, it was

found earlier in [23] that solutions of second order PDEs do not promise numerical stability in all cases. In solving the second order PDE by Zernike polynomials of a particular order (m, n) , if the terms of order higher than n are neglected in deriving operational matrices, the obtained solution is far from the actual one. By including these higher order terms as projections on the space generated by the lower order terms (see (3.8)), the solution of the second order PDE is comparable with the true solution, and accuracy in the first order case is vastly improved.

In solving PDEs using IOM and block pulse functions, simple recursive methods could be developed in [23] due to the disjoint nature of the block pulse functions. Unfortunately, this cannot be done with other orthogonal polynomials extensively used in [10] including the ones used here. The integration operational matrices for double integration, E_{Drr_0} and $E_{D\phi\phi_0}$, are computed as $E_{rr_0}^2$ and $E_{\phi\phi_0}^2$ respectively, however, this leads to an accumulation of errors at each stage of the integration process. This can be improved by developing the IOM in the final stage known as OSOMRI as developed in [23]. However, the effort does not seem worthwhile because the improvement occurs in the higher order of decimal places and in many practical cases the solution without OSOMRI may serve the purpose. Another thing to note is that the second order PDEs solved here have boundary conditions that are discontinuous at two points on the circle. Due to these discontinuities, the solution shows oscillations known as Gibbs-Wilbraham phenomenon as is evident from the discontinuity of the contour lines of IOM solutions in contrast with those of the exact analytical solution in Figure 4. In Example 3.1, a Laplace equation is solved *with discontinuous Dirichlet and Neumann BCs, and as these discontinuous functions cannot be defined at some of the Chebyshev or Gauss-Lobatto points, the much acclaimed pseudo-spectral methods are not directly applicable to such problems.* For the purpose of demonstration of our method, examples selected are simple in nature.

Our prime objective is to highlight how Zernike polynomials can be directly applied to solve PDEs with discontinuous boundary conditions. There are other methods to numerically solve PDEs as outlined in Section 1 and depending on the context and situation of the physical problems one may select an appropriate method for which the proposed approach is offered here as a potential candidate.

An extremely important problem for future investigation is the parameter estimation of distributed parameter systems that is a challenging research area for control system engineers. In this regard, another promising area of future research is to use Zernike polynomials in rectangular coordinates to solve PDEs with rectangular boundaries and conversely to estimate the parameters in such regions if the input and the response are known.

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