VARIETIES OF COMPLETELY DECOMPOSABLE FORMS AND THEIR SECANTS

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Abstract. This paper is devoted to the study of higher secant varieties of varieties of completely decomposable forms. The main goal is to develop methods to inductively verify the non-defectivity of such secant varieties. As an application of these methods, we will establish the existence of large families of non-defective secant varieties of "small" varieties of completely decomposable forms.

1. Introduction

In 1770, E. Waring suggested the problem of finding a positive integer s such that every positive integer can be written as the sum of at most s d^{th} power of positive integers. In 1909, this problem was solved affirmatively by Hilbert. There is a polynomial version of this problem, which asks, "What is the smallest integer s such that a general d-form in $(n+1)$ variables is expressible as the sum of s d^{th} powers of linear forms?" This problem is often called *Waring's problem for polynomials*. Waring's problem for polynomials has remained unsolved for many years, but was completed in a series of papers [3, 2, 1] by Alexander and Hirschowitz about 15 years ago. In this paper, we explore a variation of Waring's problem for polynomials, which will be described below.

Let $\Bbbk$ be an algebraically closed field of characteristic 0, let $R = \Bbbk[x_0, \ldots, x_n]$, and let R_d be the degree-d piece of R. If a d-form can be written as a product of linear forms, we say that the d-form is *completely decomposable*. Note that the completely decomposable d-forms form a spanning set for R_d. So every d-form is expressible as a linear combination of completely decomposable d-forms. The minimum number of completely decomposable d-forms that sum up to a d-form is called the *rank* of the d-form. It is worth noting that the rank of d-form with respect to completely decomposable d-forms is analogous in conception to the tensor rank, the symmetric tensor rank, and the skew-symmetric tenor rank. The problem we will discuss is to find the least positive integer s such that a general d-form has rank less than or equal to s, which can be viewed as a variant of the above-mentioned Waring problem for polynomials. In this paper, we will consider this problem from an algebro-geometric point of view.

The rank-one d-forms in R_d (i.e., the completely decomposable d-forms in R_d), as we shall see in Section 2.2, form an dn-dimensional variety of $\mathbb{P}R_d$. This variety is called the *variety of completely decomposable d-forms* and denoted by $\text{Split}_d(\mathbb{P}^n)$. A rank s d-form is then constructed as a point on a *secant $(s-1)$-plane* to $\text{Split}_d(\mathbb{P}^n)$, i.e., the linear subspace spanned by s linearly independent points of $\text{Split}_d(\mathbb{P}^n)$. The

The author is partly supported by NSF grant DMS-0901816.

Zariski closure of the locus of proper secant $(s-1)$-planes is a projective variety of $\mathbb{P}R_d$ called the s^{th} *secant variety* of $\text{Split}_d(\mathbb{P}^n)$, which we denote by $\sigma_s\left(\text{Split}_d(\mathbb{P}^n)\right)$.

By the construction of $\sigma_s\left(\text{Split}_d(\mathbb{P}^n)\right)$, we can easily see that finding the smallest positive integer s such that a general d-form has rank $\leq s$ is equivalent to finding the "generic rank" of $\mathbb{P}R_d$ with respect to $\text{Split}_d(\mathbb{P}^n)$, i.e., the smallest positive integer s such that $\sigma_s\left(\text{Split}_d(\mathbb{P}^n)\right)$ coincides with $\mathbb{P}R_d$. One can show that the dimension of $\sigma_s\left(\text{Split}_d(\mathbb{P}^n)\right)$ is expected to be $\min\left\{s \cdot (dn+1)-1, \binom{n+d}{d}-1\right\}$ simply by counting parameters. Thus $\left\lceil \binom{n+d}{d}/(dn+1)\right\rceil$ is the expected generic rank of $\mathbb{P}R_d$ with respect to $\text{Split}_d(\mathbb{P}^n)$.

As shall be discussed in a later paragraph of this section, there are secant varieties of varieties of completely decomposable forms, which do not have the expected dimension (such secant varieties are often said to be *defective*). If $\sigma_s\left(\text{Split}_d(\mathbb{P}^n)\right)$ is defective for some s, then $\mathbb{P}R_d$ could have larger generic rank with respect to $\text{Split}_d(\mathbb{P}^n)$ than expected. Therefore, classifying the defective secant varieties of varieties of completely decomposable forms is vital for solving our version of Waring's problem.

A crucial step toward the classification of defective secant varieties of varieties of completely decomposable forms is the development of theoretical tools, which can be used to establish the existence of large families of non-defective secant varieties to $\text{Split}_d(\mathbb{P}^n)$. The main goal of this paper is to develop a method to verify the non-defectivity of $\sigma_s(\text{Split}_d(\mathbb{P}^n))$ inductively. The method, which is inspired by the paper [5] by M. C. Brambilla and G. Ottaviani, is based on induction on d (please see Section 3 for more precise description) and enables us to reduce the problem of determining the non-defectivity of the secant variety of a variety of completely decomposable forms to the problem of determining of the non-defectivity of secant varieties of a finite collection of varieties of completely decomposable forms with smaller degrees. As an application of this method, in Section 4, we provide two explicit functions $s_1(d) < s_2(d)$ for $n \in \{2,3\}$ such that if $s \leq s_1(d)$ or $s \geq s_2(d)$, then $\sigma_s(\text{Split}_d(\mathbb{P}^n))$ has the expected dimension (please see Theorems 4.1 and 4.2 for more detail). If $n=2$, the functions s_1 and s_2 turn out to be $\left\lfloor \binom{2+d}{d}/(2d+1)\right\rfloor$ and $\left\lceil \binom{2+d}{d}/(2d+1)\right\rceil$ respectively, from which the following theorem immediately follows:

Theorem 1.1. *For any $d, s \geq 1$, $\sigma_s\left(\text{Split}_d(\mathbb{P}^2)\right)$ has the expected dimension. In particular, the generic rank of $\mathbb{P}R_d$ with respect to $\text{Split}_d(\mathbb{P}^2)$ is $\left\lceil \binom{2+d}{d}/\left(2d+1\right)\right\rceil$.*

Very recently, higher secant varieties of varieties of completely decomposable forms have been studied in at least two papers [4, 8]. In the next few paragraphs, we will review relevant results of these papers.

Inspired by a possible link between secant varieties of the Grassmann variety of lines in $\mathbb{P}^n$ and secant varieties of $\text{Split}_d(\mathbb{P}^n)$ suggested by R. Ehrenborg in [6], E. Arrondo and A. Bernardi explored problems of finding the dimensions of secant varieties of $\text{Split}_d(\mathbb{P}^n)$ in [4], where they showed that if $d \geq 3$ and $n \geq 3(s-1)$, then $\sigma_s(\text{Split}_d(\mathbb{P}^n))$ has the expected dimension. The results of this paper including Theorems 1.1 improve significantly their result for certain n and d. In the same paper, Arrondo and Bernard also showed that the dimension of the s^{th} secant variety of the Grassmann variety of lines in $\mathbb{P}^n$ equals that of $\sigma_s(\text{Split}_2(\mathbb{P}^n))$, which implies that $\sigma_s(\mathrm{Split}_2(\mathbb{P}^n))$ is defective unless $s = 1$ or $s \geq \lceil (n+1)/2 \rceil$. So far, these secant varieties are the only defective cases known to exist. So they suspect that if $d \neq 2$, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ has the expected dimension. Our result can thus be viewed as further evidence to support their presumption.

Y. S. Shin studied secant varieties of $\mathrm{Split}_d(\mathbb{P}^2)$ in connection with the Hilbert functions of special configurations of points in projective plane. To be more precise, in the recent paper [8], he proved that if $d \geq 3$, then $\sigma_2(\mathrm{Split}_d(\mathbb{P}^2))$ has the expected dimension by using the equivalence between finding the dimension of $\sigma_2(\mathrm{Split}_d(\mathbb{P}^2))$ and finding the value of the Hilbert function of the union of two "linear star-configurations of points of type d in projective plane" at d. Here we mean by a linear star-configuration of points of type d in projective plane a collection of $\binom{d}{2}$ points in projective plane obtained as the intersection of d distinct lines. In the same paper, he asked the question as to whether if $s \geq 3$ and $d \geq 6$, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^2))$ also has the expected dimension. Theorem 1.1 provides an affirmative answer to this question.

This paper is organized as follows: In Section 2, we will recall some basics related to secant varieties and varieties of completely decomposable forms. Section 3 will be devoted to the development of the aforementioned inductive approach. In Section 4, we apply this approach to show that the majority of secant varieties of $\mathrm{Split}_d(\mathbb{P}^n)$ for $n \in \{2, 3\}$ have the expected dimension. Finally, Section 5 shows that almost the same idea as in Section 3 leads to a method for finding the dimension of $\sigma_s(\mathrm{Split}_d(\mathbb{P}^n))$ by induction on n.

The proofs of Theorems 4.1, 4.2 and Corollary 5.2 are partially based on computations in `Macaulay2`. (`Macaulay2` is a computer algebra system developed by Dan Grayson and Mike Stillman [7] for research in algebraic geometry and commutative algebra.) All the `Macaulay2` scripts needed to make these computations are available at `http://www.webpages.uidaho.edu/~abo/programs.html`. Note that we work in characteristic $p = 32003$. Our task is, essentially, to check that a certain integer matrix has maximum rank. The `Macaulay2` scripts determine that the matrix has maximum rank modulo p. Thus the result in characteristic zero follows from the openness of the maximal rank condition.

2. Preliminaries

Throughout this paper, $\mathbb{k}$ denotes an algebraically closed field of characteristic 0. Let $R = \mathbb{k}[x_0, \ldots, x_n]$ and let R_d be the degree-d piece of R. If V is an $(n+1)$-dimensional vector space over $\mathbb{k}$, then we denote by $\mathbb{P}(V)$ (or just by $\mathbb{P}V$ or by $\mathbb{P}^n$) the projective space of lines in V passing through the origin $\mathbf{0}$, by $[v]$ the equivalence class containing $v \in V \setminus \{\mathbf{0}\}$, and by $\langle S \rangle$ the linear span of a subset S of $\mathbb{P}V$. Let X be a projective variety of $\mathbb{P}V$ and let p be a non-singular point of X. Then we denote by T_pX the projectivized tangent space to X at p and write $\widehat{T}_pX$ for the affine cone over T_pX.

2.1. **Secant varieties.** Let X be a projective variety of $\mathbb{P}V$ and let $p_1, \ldots, p_s$ be s generic points of X. If $s \leq n+1$, then $p_1, \ldots, p_s$ are linearly independent, and thus they form an $(s-1)$-dimensional linear subspace of $\mathbb{P}V$. We call such a linear subspace a *secant* $(s-1)$*-plane to* X. The s^{th} *secant variety of* X, denoted $\sigma_s(X)$, is defined to be the Zariski closure of the union of all secant $(s-1)$-planes to X.

A standard parameter count implies

$$\dim \sigma_s(X) \leq \min\{s \cdot (\dim X + 1), n\} - 1. \tag{2.1}$$

We say that $\sigma_s(X)$ has the *expected dimension* if Equality (2.1) actually holds. Otherwise $\sigma_s(X)$ is said to be *defective*. The variety X itself is said to be *defective* if $\sigma_s(X)$ is defective for some positive integer s.

One way of finding the dimension of $\sigma_s(X)$ is to compute the dimension of the projectivized tangent space $T_q\sigma_s(X)$ to $\sigma_s(X)$ at a generic point q. The so-called *Terracini lemma* allows one to describe $T_q\sigma_s(X)$ in terms of a collection of tangent spaces to X at s generic points.

Theorem 2.1 (Terracini's lemma)**.** *Let X be a projective variety of $\mathbb{P}V$, let $p_1, \ldots, p_s$ be s generic points of X, and let q be a generic point of $\langle p_1, \ldots, p_s\rangle$. Then $T_q\sigma_s(X) = \langle T_{p_1}X, \ldots, T_{p_s}X\rangle$.*

Remark 2.2. Let X be as above and let $a = (n+1)/(\dim X + 1)$. Then it follows immediately from Terracini's lemma that if $s \leq \lfloor a \rfloor$ and if $\sigma_s(X)$ is non-defective, then $\sigma_{s'}(X)$ is also non-defective for every $s' \leq s$, while if $s \geq \lceil a \rceil$ and if $\sigma_s(X)$ is non-defective, then $\sigma_{s'}(X)$ is also non-defective for every $s' \geq s$. In particular, to prove that $\sigma_s(X)$ is non-defective for every $s \geq 1$, it is equivalent to prove that $\sigma_s(X)$ is non-defective for every s with $\lfloor a \rfloor \leq s \leq \lceil a \rceil$.

2.2. **Varieties of completely decomposable forms.** For a given positive integer d, let $\mathrm{Split}_d(\mathbb{P}R_1) = \{[F] \in \mathbb{P}R_d \mid F = L_1 \cdots L_d \text{ for some } L_i \in R_1 \setminus \{0\}\}$. Note that $\mathrm{Split}_d(\mathbb{P}R_1)$ can be interpreted as the image of d copies of $\mathbb{P}R_1$ to $\mathbb{P}R_d$ under the morphism $\varphi : (\mathbb{P}R_1)^d \to \mathbb{P}R_d$ given by $\varphi([L_1], \ldots, [L_d]) = [L_1 \cdots L_d]$. Moreover, the fiber of φ over each point is finite, and so $\mathrm{Split}_d(\mathbb{P}R_1)$ is a projective variety of $\mathbb{P}R_d$ of dimension nd. We call $\mathrm{Split}_d(\mathbb{P}R_1)$ the *variety of completely decomposable d-forms*. The "hat" accent over $\mathrm{Split}_d(\mathbb{P}R_1)$, $\widehat{\mathrm{Split}}_d(\mathbb{P}R_1)$, denotes the affine cone over $\mathrm{Split}_d(\mathbb{P}R_1)$, or equivalently, the set of completely decomposable d-forms plus 0.

The proposition below is well known to experts. We however include a proof for the sake of completeness.

Proposition 2.3. *Let $L_1, \ldots, L_d \in R_1 \setminus \{0\}$ be generic linear forms and let $F = \prod_{i=1}^d L_i \in R_d$. Then $T_{[F]}\mathrm{Split}_d(\mathbb{P}R_1) = \sum_{i=1}^d L_1 \cdots L_{i-1}R_1L_{i+1}\cdots L_d$.*

Proof. For each $i \in \{1, \ldots, d\}$, let $M_i \in R_1 \setminus \{0\}$ be arbitrary. Consider the parametric curve

$$\prod_{i=1}^d (L_i + tM_i) \quad = \quad \prod_{i=1}^d L_i + \left(\sum_{i=1}^d L_1 \cdots L_{i-1}M_iL_{i+1}\cdots L_d\right)t + \cdots .$$

The limit of the parametric curve as $t \to 0$ is $\sum_{i=1}^d L_1 \cdots L_{i-1}M_iL_{i+1}\cdots L_d$. We thus obtain the desired equality. □

Remark 2.4. Let s and d be fixed positive integers. For $i \in \{1, \ldots, s\}$, let $L_1^{(i)}, \ldots, L_d^{(i)} \in R_1$ be generic linear forms and let $F_i = \prod_{j=1}^d L_j \in R_d$. Then a hyperplane of $\mathbb{P}R_d$ contains $T_{[F_i]}\mathrm{Split}_d(\mathbb{P}R_1)$ if and only if it contains the double point $2[F_i]$ supported at $[F_i]$, or the zero-dimensional closed scheme of $\mathbb{P}R_d$ obtained by $\mathcal{I}^2_{[F_i]}$. Thus it follows from Theorem 2.1 that if $[F] \in \langle [F_1], \ldots, [F_s]\rangle$ is generic, then the dimension of the tangent space to $\sigma_s(\mathrm{Split}_d(\mathbb{P}R_1))$ at $[F]$ equals the value of the Hilbert function $h_{\mathbb{P}R_d}(Z,\cdot)$ of $Z=\{2[F_1],\ldots,2[F_s]\}$ at 1, i.e.,

$$\begin{aligned}\dim \widehat{T}_{[F]}\sigma_s(\mathrm{Split}_d(\mathbb{P}R_1)) &= h_{\mathbb{P}R_d}(Z,1)\\ &= \binom{n+d}{n} - \dim H^0(\mathcal{I}_Z(1)).\end{aligned}$$

So we have the inequality

$$\dim H^0(\mathcal{I}_Z(1)) \geq \min\{s\cdot(nd+1), N(d)\},$$

where equality holds if and only if $\sigma_s(\mathrm{Split}_d(\mathbb{P}R_1))$ has the expected dimension.

3. Induction

In this section, we will develop an inductive approach to examine the non-defectivity of secant varieties of varieties of completely decomposable forms.

Let $n,d\in\mathbb{N}$ and let $\ell\in\mathbb{N}$ with $d\geq\ell$ and $\ell^{n-1}/(n\cdot n!)\in\mathbb{N}$. We write t and r for the quotient and the remainder when dividing d by ℓ respectively. We sometimes regard d as a linear function of t. Let $s(t)\in\mathbb{Z}[t]$ with $\deg(s)=n-1$ such that the leading coefficient $\mathrm{lc}(s)$ of $s(t)$ is $\ell^{n-1}/(n\cdot n!)$. For each $i\in\{0,1,\ldots,n\}$, we define an $f_{i,s}(t)\in\mathbb{Z}[t]$ inductively as follows:

$$\begin{cases} f_{0,s}(t) = s(t), \\ f_{i+1,s}(t) = f_{i,s}(t) - f_{i,s}(t-1) \quad \text{if } i\geq 0.\end{cases} \tag{3.1}$$

Then $\deg(f_{i,s})=n-i-1$. Moreover, $\mathrm{lc}(f_{i+1,s})=(n-i+1)\cdot\mathrm{lc}(f_{i,s})$ for each $i\in\{1,\ldots,n-1\}$. In particular, $\mathrm{lc}(f_{n-1,s})=(n-1)!\cdot\mathrm{lc}(f_{0,s})=\ell^{n-1}/n^2$ and $f_{n,s}(t)=0$.

For each $i\in\{0,\ldots,n\}$, let

$$\begin{aligned} g_i(t) &= \sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}\binom{n+d-\ell j}{n}\\ &= \sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}\binom{n+\ell(t-j)+r}{n},\end{aligned}$$

let $h_{i,s}(t)=n\ell i f_{i-1,s}(t-1)+f_{i,s}(t)(nd+1)=n\ell i f_{i-1,s}(t-1)+f_{i,s}(t)(n\cdot(\ell t+r)+1)$, and let $a_{i,s}(t)=g_i(t)+h_{i,s}(t)$. For simplicity, we write f_i, h_i, and a_i for $f_{i,s}$, $h_{i,s}$, and $a_{i,s}$ respectively if s is understood.

A straightforward calculation shows

$$g_{i+1}(t)+g_i(t-1)-g_i(t)=\binom{n+d-\ell}{n}.$$

Additionally, it follows immediately from (3.1) that

$$h_{i+1}(t)+h_i(t-1)-h_i(t)=0.$$

We thus obtain the following relation:

$$a_{i+1}(t)+a_i(t-1)-a_i(t)=\binom{n+d-\ell}{n}. \tag{3.2}$$

Definition 3.1. Let $G_0 = 0$ and let $G_1, \dots, G_n \in \widehat{\mathrm{Split}}_\ell(\mathbb{P}R_1)$ be n generic completely decomposable ℓ-forms. Fix an $i \in \{0, \dots, n\}$. Let $F_j \in \widehat{\mathrm{Split}}_d(\mathbb{P}R_1)$ be a generic completely decomposable d-form for each $j \in \{1, \dots, f_i(t)\}$, let $F_j^{(i)} \in \widehat{\mathrm{Split}}_{d-\ell}(\mathbb{P}R_1)$ be a generic completely decomposable $(d-\ell)$-form for each $j \in \{1, \dots, f_{i-1}(t-1)\}$, and let $A_i(n,d)$ be the following subspace of R_d:

$$\sum_{j=0}^{i} G_j R_{d-\ell} + \sum_{j=1}^{i} \sum_{k=1}^{f_{i-1}(t-1)} \widehat{T}_{\left[G_j F_k^{(j)}\right]} \mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_i(t)} \widehat{T}_{[F_j]} \mathrm{Split}_d(\mathbb{P}R_1).$$

For each $k \in \{1, \dots, f_{i-1}(t-1)\}$, let $F_k^{(j)} = L_{k,1}^{(j)} \cdots L_{k,d-\ell}^{(j)}$ be the factorization of $F_k^{(j)}$ into linear forms. Let $G_j = M_{j,1} \cdots M_{j,\ell}$ be the factorization of G_j into linear forms. Then

$$\begin{aligned} \widehat{T}_{\left[G_j F_k^{(j)}\right]} \mathrm{Split}_d(\mathbb{P}R_1) \quad = \quad & G_j \left(\sum_{\alpha=1}^{d-\ell} L_{k,1}^{(j)} \cdots L_{k,\alpha-1}^{(j)} R_1 L_{k,\alpha+1}^{(j)} \cdots L_{k,d-\ell}^{(j)} \right) \\ & + F_k^{(j)} \left(\sum_{\alpha=1}^{\ell} M_{j,1} \cdots M_{j,\alpha-1} R_1 M_{j,\alpha+1} \cdots M_{j,\ell} \right). \end{aligned}$$

Both the first and the second summands of the right side of the equality must contain the subspace spanned by $G_j F_k^{(j)}$, and so we get

$$\begin{aligned} \dim \left(G_j R_{d-\ell} + \widehat{T}_{\left[G_j F_k^{(j)}\right]} \mathrm{Split}_d(\mathbb{P}R_1) \right) \quad & \leq \quad \binom{n+d-\ell}{n} + (n+1)\ell - \ell \\ & = \quad \binom{n+d-\ell}{n} + n\ell. \end{aligned}$$

Furthermore, $\dim \sum_{j=1}^{i} G_j R_{d-\ell} = g_i(t)$ and $\dim \widehat{T}_{[F_j]} \mathrm{Split}_d(\mathbb{P}R_1) = nd + 1$. So if $a_i(t) \leq \binom{n+d}{d}$, then

$$\dim A_i(n,d) \leq g_i(t) + n\ell i f_{i-1}(t-1) + f_i(t)(nd+1) = a_i(t).$$

In other words,

$$\dim A_i(n,d) \leq \min \left\{ a_i(t),\ \binom{n+d}{d} \right\}. \tag{3.3}$$

We say that the statement $\mathfrak{A}_i(n,d,s)$ (or simply $\mathfrak{A}_i(n,d)$ with s understood) is *true* if Equality (3.3) holds.

Remark 3.2. If $i = 0$, then

$$A_0(n,d) = \sum_{j=1}^{f_0(t)} \widehat{T}_{[F_j]} \mathrm{Split}_d(\mathbb{P}R_1) = \sum_{j=1}^{s(t)} \widehat{T}_{[F_j]} \mathrm{Split}_d(\mathbb{P}R_1),$$

and so it follows from Terracini's lemma that "$\mathfrak{A}_0(n,d)$ is true" implies "$\sigma_{s(t)}(\mathrm{Split}_d(\mathbb{P}R_1))$ has the expected dimension".

Definition 3.3. We call the statement $\mathfrak{A}_i(n,d)$ *sub-abundant* if $a_i(t) \leq \binom{n+d}{n}$ and *super-abundant* if $a_i(t) \geq \binom{n+d}{n}$. The statement $\mathfrak{A}_i(n,d)$ is said to be *equiabundant* if $\mathfrak{A}_i(n,d)$ is both sub-abundant and super-abundant.

Proposition 3.4. *Let $d \in \mathbb{N}$ with $d \geq n\ell$. Then $\mathfrak{A}_n(n,d)$ is equiabundant.*

Proof. Note that the leading coefficient of $\binom{n+d}{n} = \binom{n+\ell t+r}{n}$ is $\ell^n/n!$. Thus

$$\begin{aligned} & a_n(t) \\ = & \sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\binom{n+\ell(t-j)+r}{n} + n^2\ell f_{n-1}(t-1) + f_n(t)(n(\ell n + r)+1) \\ = & \binom{n+\ell t+r}{n} - n!\cdot \mathrm{lc}\binom{n+\ell t+r}{n} + n^2\ell\cdot\frac{\ell^{n-1}}{n^2} \\ = & \binom{n+\ell t+r}{n} - n!\cdot\frac{\ell^n}{n!} + \ell^n \\ = & \binom{n+d}{n}. \end{aligned}$$

Therefore, $\mathfrak{A}_n(n,d)$ is equiabundant. ☐

Remark 3.5. For each $i \in \{1,\dots,n\}$, let us, keeping the same notation as in Definition 3.1, write P for $\bigcup_{j=1}^{i}\mathbb{P}(G_jR_{d-\ell})$ and Z for the following set of double points:

$$\{2[F_j] \mid 1 \le j \le f_i(t)\} \cup \left\{ 2\left[G_jF_k^{(j)}\right] \,\middle|\, 1 \le j \le i,\ 1 \le k \le f_{i-1}(t-1)\right\}.$$

Note that a hyperplane in $\mathbb{P}R_d$ contains $\mathbb{P}A_i(n,d)$ precisely when it contains $Z \cup P$. Thus

$$\dim A_i(n,d) = \binom{n+d}{n} - \dim H^0(\mathcal{I}_{Z\cup P}(1)),$$

and hence

$$\dim H^0(\mathcal{I}_{Z\cup P}(1)) \ge \min\left\{\binom{n+d}{n} - a_i(t),\ 0\right\}. \tag{3.4}$$

Therefore, $\mathfrak{A}_i(n,d)$ is true if and only if Equality (3.4) holds.

Remark 3.6. One can directly check the truth of $\mathfrak{A}_i(n,d)$ for given $n,d \in \mathbb{N}$ with a computer algebra system as follows: Select $f_i(t)$ d-forms $F_1,\dots,F_{f_i(t)}$, $if_{i-1}(t-1)$ $(d-\ell)$-forms $F_1^{(j)},\dots,F_{f_{i-1}(t-1)}^{(j)}$ for each $j \in \{1,\dots,i\}$, and $i+1$ ℓ-forms $G_0,\dots,G_i$ as indicated in Definition 3.1 using a random command. Form the vector space

$$A_i(n,d) = \sum_{j=0}^{i} G_jR_{d-\ell} + \sum_{j=1}^{i}\sum_{k=1}^{f_{i-1}(t)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_i(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1),$$

and then consider the ideal of $\Bbbk[x_0,\dots,x_n]$ generated by $A_i(n,d)$. The statement $\mathfrak{A}_i(n,d)$ is true if the ideal is minimally generated by the same number of d-forms as $\min\left\{a_i(t),\ \binom{n+d}{n}\right\}$. As stated above, this can be done with the aid of a computer algebra system such as `Macaulay2`. In the rest of this remark, we illustrate the use of the `Macaulay2` command called `statementAi` the author wrote for the above-mentioned computation.

The `statementAi` command takes four integers n, d, i, ℓ and a pair of two integers $f_i(t)$ and $f_{i-1}(t-1)$. It computes the ideal generated by $A_i(n,d)$ and minimizes its generators. Below we discuss an explicit example to demonstrate more precisely how the command works.

Let $n = 2$, let $\ell = 4$, and let $d = 4t+2$ with $t \ge 0$. Define $s_1(t) \in \mathbb{Z}[t]$ by $s_1(t) = t+1$. By definition, $f_0(t) = t+1$, $f_1(t) = 1$, and $f_2(t) = 0$. We now show that $\mathfrak{A}_1(2,6)$ is true. If $d = 6$, then $t = 1$, so we have $f_0(t-1) = f_0(0) = 1$. Thus the input is $(n, d, i, \ell, \{f_1(t), f_0(t-1)\}) = (2, 6, 1, 4, \{1, 1\})$. We now compute the ideal as follows:

```
i2 : I = statementAi(2,6,1,4,{1,1});

                  ZZ
o2 : Ideal of -----[x , x , x ]
                32003  0   1   2
```

Considering the set of generators for I as an R-linear map between two free R-modules, we can compute the number of generators as the rank of the source free R-module.

```
i3 : rank source gens I

o3 = 27
```

Note that the generating set for the ideal `I` has been already minimized. Since

$$\begin{aligned}\min\left\{a_1(2,6), \binom{2+6}{2}\right\} &= \min\left\{\binom{2+6-4}{2} + 2\cdot 4\cdot 1\cdot 1 + 1\cdot(2\cdot 6+1), 28\right\}\\ &= 27,\end{aligned}$$

this means that $A_1(2,6)$ has the expected dimension, and hence $\mathfrak{A}_1(2,6)$ is true.

Proposition 3.7. *Let $d \in \mathbb{N}$ with $d \geq 2\ell$. Suppose that $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n, d-\ell)$ have the same abundancy. If $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$ are both true, then $\mathfrak{A}_i(n,d)$ is also true and has the same abundancy as $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$.*

Proof. It is an immediate consequence of (3.2) that $\mathfrak{A}_i(n,d)$ has the same abundancy as $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$.

With the same notation as in Definition 3.1, we assume that G_{i+1} divides $F_{f_{i+1}(t)+1}, \ldots, F_{f_i(t)}$ and $F^{(j)}_{f_i(t-1)+1}, \ldots, F^{(j)}_{f_{i-1}(t-1)}$ for each $j \in \{1, \ldots, i\}$. We denote by $F_j^{(i+1)}$ the completely decomposable $(d-\ell)$-form such that $F_{f_{i+1}(t)+j} = G_{i+1}F_j^{(i+1)}$. Assume that $F_j^{(i+1)} \in \widehat{\mathrm{Split}}_{d-\ell}(\mathbb{P}R_1)$ is generic for each $j \in \{1, \ldots, f_1(t-1)\}$.

Let $P_{i+1} = \mathbb{P}(G_{i+1}R_{d-\ell})$. Keeping the same notation as in Remark 3.5, we then have the following short exact sequence of sheaves:

$$0 \to \mathcal{I}_{Z\cup P\cup P_{i+1}}(1) \to \mathcal{I}_{Z\cup P}(1) \to \mathcal{I}_{(Z\cup P)\cap P_{i+1}, P_{i+1}}(1) \to 0.$$

So, by taking the cohomology, we obtain the inequality

$$\dim H^0(\mathcal{I}_{Z\cup P}(1)) \leq \dim H^0(\mathcal{I}_{(Z\cup P)\cap P_{i+1}, P_{i+1}}(1)) + \dim H^0(\mathcal{I}_{Z\cup P\cup P_{i+1}}(1)). \tag{3.5}$$

By definition, $F_{f_{i+1}(t)+1}, \ldots, F_{f_i(t)}$ lie in $G_{i+1}R_{d-\ell}$, while $G_jF^{(j)}_{f_i(t-1)+1}, \ldots, G_jF^{(j)}_{f_{i-1}(t-1)}$ lie in $G_jG_{i+1}R_{d-2\ell}$. Hence it follows immediately from Proposition 2.3 that $\widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1)$ is contained in the span of

$G_jR_{d-\ell}$ and $G_{i+1}R_{d-\ell}$. Also,

$$\begin{aligned}
&\sum_{j=0}^{i+1} G_jR_{d-\ell} + \sum_{j=1}^{i}\sum_{k=1}^{f_{i-1}(t-1)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_i(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1)\\
= \;& \sum_{j=0}^{i+1} G_jR_{d-\ell}\\
&+\sum_{j=1}^{i}\left(\sum_{k=f_i(t-1)+1}^{f_{i-1}(t-1)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{k=1}^{f_i(t-1)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1)\right)\\
&+\sum_{j=f_{i+1}(t)+1}^{f_i(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_{i+1}(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1)\\
= \;& \sum_{j=0}^{i+1} G_jR_{d-\ell} + \sum_{j=1}^{i}\sum_{k=1}^{f_i(t-1)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1)\\
&+\sum_{k=1}^{f_i(t-1)} \widehat{T}_{\left[G_{i+1}F_k^{(i+1)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_{i+1}(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1)\\
= \;& \sum_{j=0}^{i+1} G_jR_{d-\ell} + \sum_{j=1}^{i+1}\sum_{k=1}^{f_i(t-1)} \widehat{T}_{\left[G_jF_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_{i+1}(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1)\\
= \;& A_{i+1}(n,d).
\end{aligned}$$

Therefore,

$$\dim H^0(\mathcal{I}_{Z\cup P\cup P_{i+1}}(1)) = \binom{n+d}{n} - \dim A_{i+1}(n,d).$$

Since both $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$ are true by assumption, $\dim A_{i+1}(n,d) = \min\left\{\binom{n+d}{n} - a_{i+1}(t),\ 0\right\}$ and $\dim A_i(n,d-\ell) = \min\left\{\binom{n+d-\ell}{n} - a_i(t-1),\ 0\right\}$. Recall that $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$ have the same abundancy. If $\mathfrak{A}_{i+1}(n,d)$ and $\mathfrak{A}_i(n,d-\ell)$ are sub-abundant, then

$$\begin{aligned}
&\dim H^0(\mathcal{I}_{(Z\cup P)\cap P_{i+1},P_{i+1}}(1)) + \dim H^0(\mathcal{I}_{Z\cup P\cup P_{i+1}}(1))\\
= \;& \binom{n+d}{n} - a_{i+1}(t) + \binom{n+d-\ell}{n} - a_i(t-1)\\
= \;& \binom{n+d}{n} - a_i(t).
\end{aligned}$$

Likewise, if they are super-abundant, then

$$\dim H^0(\mathcal{I}_{(Z\cup P)\cap P_{i+1},P_{i+1}}(1)) + \dim H^0(\mathcal{I}_{Z\cup P\cup P_{i+1}}(1)) = 0 + 0 = 0.$$

Thus it follows from (3.4) and (3.5) that

$$\dim H^0(\mathcal{I}_{Z\cup P}(1)) = \min\left\{\binom{n+d}{n} - a_i(t),\ 0\right\},$$

and hence $\mathfrak{A}_i(n,d)$ is true. $\square$

Proposition 3.8. *Let $d\in\mathbb{N}$ with $d\geq n\ell+1$. If $\mathfrak{A}_n(n,d_0)$ is true for some $d_0\in\mathbb{N}$ with $n\ell+1\leq d_0<d$, then so is $\mathfrak{A}_n(n,d)$.*

Proof. Let $K \in \widehat{\mathrm{Split}}_{d-d_0}(\mathbb{P}R_1)$ be a generic completely decomposable $(d-d_0)$-form. Keeping the same notation as in Definition 3.1, we assume that $F_1^{(i)}, \dots, F_{f_{n-1}(t)}^{(i)} \in \widehat{\mathrm{Split}}_{d-\ell}(\mathbb{P}R_1)$ are divisible by K for each $i \in \{1, \dots, n\}$. Let $H_1^{(i)}, \dots, H_{f_{n-1}(t)}^{(i)} \in \widehat{\mathrm{Split}}_{d-\ell-d_0}(\mathbb{P}R_1)$ be the $(d-\ell-d_0)$-forms such that $F_j^{(i)} = KH_j^{(i)}$. By definition, we assume that $H_1^{(i)}, \dots, H_{f_{n-1}(t)}^{(i)} \in \widehat{\mathrm{Split}}_{d-\ell-d_0}(\mathbb{P}R_1)$ be generic. Let $Z = \left\{ 2\left[G_iF_j^{(i)}\right] \;\middle|\; 1 \le i \le n, 1 \le j \le f_{n-1}(t) \right\}$.

Denote $\bigcup_{i=1}^n \mathbb{P}(G_iR_{d-\ell})$ and $\mathbb{P}(KR_{d-d_0})$ by P and P' respectively. Then we get a short exact sequence of sheaves

$$0 \to \mathcal{I}_{Z\cup P\cup P'}(1) \to \mathcal{I}_{Z\cup P}(1) \to \mathcal{I}_{(Z\cup P)\cap P', P'}(1) \to 0,$$

from which we obtain an exact sequence

$$0 \to H^0(\mathcal{I}_{Z\cup P\cup P'}(1)) \to H^0(\mathcal{I}_{Z\cup P}(1)) \to H^0(\mathcal{I}_{(Z\cup P)\cap P', P'}(1)).$$

We have shown that $\mathfrak{A}_n(n,d)$ is equiabundant, and thus all we have to do is prove that $H^0(\mathcal{I}_{Z\cup P}(1)) = 0$. By the assumption that $\mathfrak{A}_i(n, d-\ell)$ is true, the cohomology group $H^0(\mathcal{I}_{(Z\cup P)\cap P', P'}(1))$ vanishes. It is, therefore, sufficient to show that $H^0(\mathcal{I}_{Z\cup P\cup P'}(1)) = 0$.

By the choice of K,

$$\left(\sum_{i=1}^n G_iR_{d-\ell}\right) \cap KR_{d-d_o} = \sum_{i=1}^n \left(G_iR_{d-\ell} \cap KR_{d-d_0}\right) = \sum_{i=1}^n G_iKR_{d-\ell-d_0}.$$

Since $\dim \sum_{i=1}^n G_iKR_{d-\ell-d_0} = \sum_{i=1}^n (-1)^{i-1}\binom{n}{i}\binom{n+d-\ell i-d_0}{n}$ and $d = \ell t + r$ with $r \in \{0, \dots, \ell-1\}$, we have

$$\begin{aligned}
& \dim\left(\sum_{i=1}^n G_iR_{d-\ell} + KR_{d-d_0}\right) \\
= \;& \dim \sum_{i=1}^n G_iR_{d-\ell} + \dim KR_{d-d_0} - \dim\left(\sum_{i=1}^n G_iR_{d-\ell}\right) \cap KR_{d-d_o} \\
= \;& \sum_{i=1}^n (-1)^{i-1}\binom{n}{i}\binom{n+\ell(t-i)+r}{n} + \binom{n+\ell t+r-d_0}{n} \\
& -\sum_{i=1}^n (-1)^{i-1}\binom{n}{i}\binom{n+\ell(t-i)+r-d_0}{n} \\
= \;& \binom{n+\ell t+r}{n} - \mathrm{lc}\binom{n+\ell t+r}{n} + \mathrm{lc}\binom{n+\ell(t-1)+r-d_0}{n} \\
= \;& \binom{n+d}{n}.
\end{aligned}$$

This means that $H^0(\mathcal{I}_{Z\cup P\cup P'}(1)) = 0$, which completes the proof. □

The following corollary is an immediate consequence of Propositions 3.7 and 3.8:

Corollary 3.9. *Let $d_0 \in \mathbb{N}$ with $d_0 \ge n\ell + 1$. Suppose that the following two conditions hold:*

(i) *$\mathfrak{A}_n(n, d_0)$ is true.*

(ii) $\mathfrak{A}_i(n, d_0 - \ell(n-i) - j)$ *are true and have the same abundancy for all* $i \in \{0, \dots, n-1\}$ *and for all* $j \in \{1, \dots, \ell\}$.

Then $\mathfrak{A}_i(n, d)$ *is true for each* $i \in \{0, \dots, n\}$ *and for each* $d \geq d_0 - \ell(n-i)$.

Remark 3.10. By the definition of $a_i(t)$, it is straightforward to show that if d_0 is sufficiently large, then $\mathfrak{A}_i(n, d_0 - \ell(n-i) - j)$ have the same abundancy for all $i \in \{0, \dots, n-1\}$ and for all $j \in \{1, \dots, \ell\}$.

4. Secant dimensions of $\mathrm{Split}_d(\mathbb{P}^n)$ with small n

In this section, we discuss secant varieties of $\mathrm{Split}_d(\mathbb{P}^n)$ with $n \in \{2, 3\}$. The first goal is to show the non-defectivity of $\mathrm{Split}_d(\mathbb{P}^2)$ as an application of the method discussed in Section 3. More precisely, we will use Corollary 3.9 to prove

$$\dim \sigma_s(\mathrm{Split}_d(\mathbb{P}^2)) = \min\left\{ s \cdot (2d+1),\ \binom{2+d}{2} \right\} - 1$$

for every s and d. The second goal is to establish two functions $s_1(d)$ and $s_2(d)$ of d with $s_1(d) < s_2(d)$ such that $\sigma_s(\mathrm{Split}_d(\mathbb{P}^3))$ has the expected dimension for $s \leq s_1(d)$ or $s \geq s_2(d)$.

Theorem 4.1. *For each* $d, s \in \mathbb{N}$, $\sigma_s(\mathrm{Split}_d(\mathbb{P}^2))$ *has the expected dimension.*

Proof. Note that $\sigma_1(\mathrm{Split}_d(\mathbb{P}^2))$ is trivially non-defective. Additionally, in his recent paper [8], Y. S. Shin has proved that $\sigma_2(\mathrm{Split}_d(\mathbb{P}^2))$ has the expected dimension for every positive integer d. Since $2(2d+1) \geq \binom{2+d}{2}$ if and only if $d \leq 5$, it follows from Shin's result that $\sigma_s(\mathrm{Split}_d(\mathbb{P}^2))$ is not defective for every $s \geq 1$ if $d \leq 5$. Our task is therefore to establish the non-defectivity of $\mathrm{Split}_d(\mathbb{P}^2)$ for every $d \geq 6$.

Let $n = 2$. Then $\ell = 4$ is the least positive integer satisfying the desired numeric condition given at the beginning of Section 3, i.e., $\ell^{n-1}/(n \cdot n!) \in \mathbb{N}$. Let t and r be the quotient and the remainder in the division of d by 4 respectively. Define $s_1(t) \in \mathbb{Z}[t]$ by

$$s_1(t) = \begin{cases} t & \text{if } r \in \{0, 1\}, \\ t+1 & \text{if } r \in \{2, 3\} \end{cases}$$

and $s_2(t) \in \mathbb{Z}[t]$ by $s_2(t) = s_1(t) + 1$. It is not very hard to see that if $k \geq 1$, then

$$s_1(t) = \left\lfloor \frac{\binom{2+d}{d}}{2d+1} \right\rfloor \text{ and } s_2(t) = \left\lceil \frac{\binom{2+d}{d}}{2d+1} \right\rceil. \tag{4.1}$$

By Remark 2.2, in order to prove the theorem, it suffices to show the non-defectivity of $\sigma_{s_1(t)}(\mathrm{Split}_d(\mathbb{P}^2))$ and $\sigma_{s_2(t)}(\mathrm{Split}_d(\mathbb{P}^2))$ for each $d \geq 6$ (or equivalently, the truth of $\mathfrak{A}_0(2, d, s_1)$ and $\mathfrak{A}_0(2, d, s_2)$ for $d \geq 6$).

For each $j \in \{1, 2\}$, let $f_{i,s_j}(t) \in \mathbb{Z}[t]$, $i \in \{-1, \dots, 2\}$, be defined as indicated in Section 3. To be more precise,

$$\begin{cases} f_{0,s_j}(t) & = & s_j(t), \\ f_{i+1,s_j}(t) & = & f_{i,s_j}(t) - f_{i,s_j}(t-1) \quad \text{if } 0 \leq i \leq 1. \end{cases}$$

Then $f_{1,s_j}(t) = s_j(t) - s_j(t-1) = 1$ and $f_{2,s_j}(t) = 0$ for each $j \in \{1, 2\}$. Let $a_{i,s_j}(t)$ be defined as at the beginning of Section 3 for each $i \in \{0, 1, 2\}$. By (4.1), $a_{0,s_1}(t) < \binom{2+d}{d}$ and $a_{0,s_2}(t) > \binom{2+d}{d}$.

Recall that

$$a_{1,s_j}(t) = \binom{2 + 4t + r - 4}{2} + 2 \cdot 4 \cdot 1 \cdot f_{0,s_j}(t-1) + f_{1,s_j}(t)(2 \cdot (4t+r) + 1).$$

Thus

$$\binom{2+4t+r}{2} - a_{1,s_j}(t) = \begin{cases} 2r+5 & \text{if } (j,r) \in \{(1,0),(1,1)\}, \\ 2r-3 & \text{if } (j,r) \in \{(1,2),(1,3),(2,0),(2,1)\}, \\ 2r-11 & \text{if } (j,r) \in \{(2,2),(2,3)\}. \end{cases}$$

Therefore, if $d_0 = 14$ (or bigger), then $\mathfrak{A}_i(2, d_0-4i-k, s_j)$ have the same abundancy for all $i \in \{0,1\}$ and $k \in \{1,\dots,4\}$. As indicated in Remark 3.6, one can verify, with the aid of a computer algebra system, that $\mathfrak{A}_1(2, d_0-4\cdot 1-k, s_j)$ and $\mathfrak{A}_0(2, d_0-4\cdot 2-k, s_j)$, $k \in \{1,\dots,4\}$, as well as $\mathfrak{A}_2(2, d_0, s_j)$ are all true, and so Theorem 4.1 follows immediately from Corollary 3.9. □

Let $n = 3$ and let $\ell = 6$. Then ℓ is the smallest positive integer such that $\ell^{n-1}/(n \cdot n!) \in \mathbb{N}$. Let $d \in \mathbb{N}$. We write t and r for the quotient and the remainder when dividing d by 6. Define $s_1(t), s_2(t) \in \mathbb{Z}[t]$ by

$$(4.2) \quad s_1(t) = \begin{cases} 2t^2+t+1 & \text{if } r=0 \\ 2t^2+2t & \text{if } r=1 \\ 2t^2+3t+1 & \text{if } r=2 \\ 2t^2+3t+1 & \text{if } r=3 \\ 2t^2+4t+2 & \text{if } r=4 \\ 2t^2+5t+3 & \text{if } r=5 \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} 2t^2+2t+1 & \text{if } r=0 \\ 2t^2+3t+2 & \text{if } r=1 \\ 2t^2+4t+2 & \text{if } r=2 \\ 2t^2+4t+2 & \text{if } r=3 \\ 2t^2+5t+3 & \text{if } r=4 \\ 2t^2+6t+4 & \text{if } r=5 \end{cases}$$

respectively.

Theorem 4.2. *If $s \le s_1(t)$ or $s \ge s_2(t)$, then $\sigma_s(\mathrm{Split}_d(\mathbb{P}^3))$ is non-defective for every $d \in \mathbb{N}$.*

Proof. A simple calculation shows that, for each $d \in \mathbb{N}$, the statements $\mathfrak{A}_0(3, d, s_1)$ and $\mathfrak{A}_0(3, d, s_2)$ are sub-abundant and super-abundant respectively. Thus, by Remark 2.2, it is sufficient to show that both $\sigma_{s_2(t)}(\mathrm{Split}_d(\mathbb{P}^3)$ and $\sigma_{s_1(t)}(\mathrm{Split}_d(\mathbb{P}^3)$ have the expected dimension for every $d \in \mathbb{N}$, or equivalently, $\mathfrak{A}_0(3, d, s_1)$ and $\mathfrak{A}_0(3, d, s_2)$ are true for every $d \in \mathbb{N}$.

Let $j \in \{1,2\}$ and let $f_{i,s_j}(t) \in \mathbb{Z}[t]$, $i \in \{-1,\dots,3\}$, be defined as indicated in Section 3. Then $f_{2,s_j}(t) = \ell^2/(n\cdot n!) \cdot \mathrm{lc}(f_{0,s_j}) = 6^2/(3\cdot 3!)\cdot 2 = 2\cdot 2 = 4$. Also,

$$f_{1,s_1}(t) = \begin{cases} 4t-1 & \text{if } r=0 \\ 4t & \text{if } r=1 \\ 4t+1 & \text{if } r=2 \\ 4t+1 & \text{if } r=3 \\ 4t+2 & \text{if } r=4 \\ 4t+3 & \text{if } r=5 \end{cases}$$

and $f_{1,s_2}(t) = f_{1,s_1}(t) + 1$. We thus obtain the following table:

r	$\binom{3+6t+r}{3} - a_{1,s_1}(t)$	$\binom{3+6t+r}{3} - a_{1,s_1}(t)$	$\binom{3+6t+r}{3} - a_{2,s_1}(t)$	$\binom{3+6t+r}{3} - a_{2,s_2}(t)$
0	$32t-24$	$-4t-7$	32	-4
1	$20t+8$	$-16t-14$	20	-16
2	$8t+4$	$-28t-3$	8	-28
3	$32t+10$	$-4t$	32	-4
4	$20t+9$	$-16t-14$	20	-16
5	$8t+8$	$-28t-8$	8	-28

Let $i \in \{1,2\}$, let $j \in \{1,\dots,6\}$, and let $k \in \{1,2\}$. If $d_0 = 19$ (or more), then the above table implies that $\mathfrak{A}_i(n, d_0 - 12 - j, s_k)$ has the same abundancy as $\mathfrak{A}_0(3, d_0 - 18, s_k)$ (note that if $r = 0$, then t must be greater than 0).

Note that if $d = 1$ (i.e., $t = 1$ and $r = 0$), then $s_1(0) = 0$, and so the statement $\mathfrak{A}_0(3,1,0)$ is trivially true. Also, $s_1(t) = 1$ precisely when $t = 0$ and $r \in \{2,3\}$. Hence $\mathfrak{A}_0(3,2,1)$ and $\mathfrak{A}_0(3,3,1)$ are also trivially true. Furthermore, $s_1(0) = 2$ if $r = 4$. Therefore, $\mathfrak{A}_0(3,4,2)$ is true by Proposition 1.8 in [4].

Let $k \in \{1,2\}$. Then one can computationally check the truth of the following statements in the way as suggested in Remark 3.6:

- $\mathfrak{A}_3(3, d_0, s_k)$,
- $\mathfrak{A}_i(n, d_0 - 6(3-i) - j, s_k)$ with $i \in \{1,2\}$ and $j \in \{1,\dots,6\}$,
- $\mathfrak{A}_0(3,5,3)$ and $\mathfrak{A}_0(3,6,4)$.

Thus Theorem 4.2 is a consequence of Corollary 3.9. □

5. A closing comment

The purpose of this section is to see that the idea discussed in Section 3 can be naturally adapted to develop an inductive approach that allows one to show the non-defectivity of secant varieties of varieties of completely decomposable forms by verifying the non-defectivity of secant varieties of varieties of completely decomposable forms in a smaller number of variables. The key fact is that $\dim R_d = \binom{n+d}{n} = \binom{n+d}{d}$ and $\dim \mathrm{Split}_d(\mathbb{P}R_1) = dn+1$ are symmetric polynomials of $\mathbb{Q}[n,d]$. Thus the basic idea to develop such an inductive approach is essentially just replace the roles of d and n in the argument of Section 3!

Let $n, d \in \mathbb{N}$, let $R = \Bbbk[x_0, \cdots, x_n]$, and let R_d be the homogeneous component of R of degree d. Let ℓ be a positive integer satisfying $\ell^{d-1}/(d \cdot d!) \in \mathbb{N}$. We denote by t and r the quotient and remainder when we divide n by ℓ. Let $s \in \mathbb{Z}[t]$ whose leading coefficient is $\ell^{d-1}/(d \cdot d!)$. Define a collection $\{f_{i,s}\}_{-1 \le i \le d}$ of polynomials in the same way as in Section 3. For each $i \in \{0,\dots,d\}$, replace the roles of n and d in g_i and $h_{i,s}$ given in Section 3. Again, denote these polynomials by g_i and $h_{i,s}$ respectively, i.e., $g_i(t) = \sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}\binom{\ell(t-j)+r+d}{d}$ and $h_{i,s}(t) = d\ell i f_{i-1,s}(t-1) + f_{i,s}(t)\,((\ell t + r)\cdot d + 1)$.

Let $i \in \{0,\dots,d\}$, let V_i be a generic $(n+1-\ell)$-dimensional subspace of R_1 and let V_i' be a generic subspace of R_1 such that $R_1 = V_i \oplus V_i'$. Then, by the choice of the V_i's, $\dim \sum_{j=1}^{i} \mathrm{Sym}_d V_j = g_i(t)$, where $\mathrm{Sym}_d V_j$ denotes the d^{th} symmetric power of V_j. Let $F_1, \dots, F_{f_{i,s}(t)} \in \widehat{\mathrm{Split}}_d(\mathbb{P}R_1)$ and let $F_1^{(j)}, \dots, F_{f_{i,s}(t-1)}^{(j)} \in \widehat{\mathrm{Split}}_d(\mathbb{P}V_j)$ for each $j \in \{1,\dots,i\}$. If $F_k^{(j)} = L_{k,1}^{(j)} \cdots L_{k,d}^{(j)}$ is the factorization of $F_k^{(j)}$ into linear forms, then

$$\begin{aligned}
&\widehat{T}_{\left[F_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1)\\
= \;& \sum_{\alpha=1}^{d} L_{k,1}^{(j)}\cdots L_{k,\alpha-1}^{(j)} R_1 L_{k,\alpha+1}^{(j)}\cdots L_{k,d}^{(j)}\\
= \;& \sum_{\alpha=1}^{d} L_{k,1}^{(j)}\cdots L_{k,\alpha-1}^{(j)} V_1 L_{k,\alpha+1}^{(j)}\cdots L_{k,d}^{(j)} \oplus \sum_{\alpha=1}^{d} L_{k,1}^{(j)}\cdots L_{k,\alpha-1}^{(j)} V_1' L_{k,\alpha+1}^{(j)}\cdots L_{k,d}^{(j)}\\
= \;& \widehat{T}_{\left[F_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}V_1) \oplus \sum_{\alpha=1}^{d} L_{k,1}^{(j)}\cdots L_{k,\alpha-1}^{(j)} V_1' L_{k,\alpha+1}^{(j)}\cdots L_{k,d}^{(j)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\dim\left(\mathbb{P}\mathrm{Sym}_d V_1 + \widehat{T}_{\left[F_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1)\right) \;&\leq\; \binom{n-\ell+d}{d} + \dim V_1'\\
&=\; \binom{n-\ell+d}{d} + d\ell.
\end{aligned}$$

Therefore, the subspace of $\mathbb{P}R_d$

$$\sum_{j=1}^{i} V_j + \sum_{j=1}^{i}\sum_{k=1}^{f_{i-1}(t-1)} \widehat{T}_{\left[F_k^{(j)}\right]}\mathrm{Split}_d(\mathbb{P}R_1) + \sum_{j=1}^{f_i(t)} \widehat{T}_{[F_j]}\mathrm{Split}_d(\mathbb{P}R_1), \tag{5.1}$$

denoted $B_i(n,d,s)$, is expected to have dimension less than or equal to $b_{i,s}(t)$ unless $B_i(n,d,s) = \mathbb{P}R_d$. We say that the statement $\mathfrak{B}_i(n,d,s)$ is true if $B_i(n,d,s)$ has dimension $\min\left\{b_{i,s}(t), \binom{n+d}{d}\right\}$.

As in Definition 3.3, we call $\mathfrak{B}_i(n,d,s)$ *sub-abundant* if $b_{i,s}(n,d) = \min\left\{b_{i,s}(t), \binom{n+d}{d}\right\}$, *super-abundant* if $\binom{n+d}{d} = \min\left\{b_{i,s}(t), \binom{n+d}{d}\right\}$, and *equiabundant* if $b_{i,s}(t), \binom{n+d}{d}$. Because of the choice of the V_i's, we have $\dim \sum_{j=1}^{d} V_j = b_{i,s}(n,d)$, which equals $\binom{n+d}{d}$, and so $\mathfrak{B}_d(n,d,s)$ is equiabundant.

Let $i \in \{0,\dots,d\}$, let $P_j = \mathbb{P}V_j$ for each $j \in \{1,\dots,i\}$, and let

$$Z = \{2[F_j] \mid 1 \leq j \leq f_{i,s}(t)\} \cup \left\{ 2\left[F_k^{(j)}\right] \;\middle|\; 1 \leq j \leq i, 1 \leq k \leq f_{i-1,s}(t)\right\}.$$

For simplicity, we write P for $\bigcup_{j=1}^{i} P_i$. Then $b_{i,s}(n,d) = \binom{n+d}{d} - \dim H^0(\mathcal{I}_{Z\cup P}(1))$.

Now assume that $F_{f_{i+1}(t)+1},\dots,F_{f_i(t)} \in V_{i+1}$ and $F^{(j)}_{f_i(t-1)+1},\dots,F^{(j)}_{f_{i-1}(t-1)} \in V_j \cap V_{i+1}$ for each $j \in \{1,\dots,i\}$. Consider the exact sequence

$$0 \to \mathcal{I}_{Z\cup P\cup P_{i+1}}(1) \to \mathcal{I}_{Z\cup P}(1) \to \mathcal{I}_{(Z\cup P)\cap P_{i+1},P_{i+1}}(1) \to 0.$$

Then, in the same way as in the proof of Proposition 3.7 (with slight modification), one can show that if $\mathfrak{B}_{i+1}(n,d,s)$ and $\mathfrak{B}_i(n-\ell,d,s)$ have the same abundancy and if they are both true, then $\mathfrak{B}_i(n,d)$ is also true and has the same abundancy as $\mathfrak{B}_{i+1}(n,d,s)$ and $\mathfrak{B}_i(n-\ell,d,s)$. We therefore obtain the following theorem:

Theorem 5.1. *Let $n_0 \in \mathbb{N}$ with $n_0 \geq d\ell + 1$. Suppose that the following two conditions hold:*

(i) *$\mathfrak{B}_d(n_0,d,s)$ is true.*

(ii) $\mathfrak{B}_i(n_0 - \ell(d-i) - j, d, s)$ *are true and have the same abundancy for all* $i \in \{0, \dots, d-1\}$ *and for all* $j \in \{1, \dots, \ell\}$.

Then $\mathfrak{B}_i(n, d, s)$ *is true for each* $i \in \{0, \dots, d\}$ *and for each* $n \geq n_0 - \ell(d-i)$.

Corollary 5.2. *Let* $d = 3$*, let* $\ell = 6$*, and let* $n \in \mathbb{N}$*. We write* t *and* q *for the quotient and the remainder when dividing* d *by* 6*. Let* $s_1(t), s_2(t) \in \mathbb{Z}[t]$ *be as defined in (4.2). If* $s \leq s_1(t)$ *or* $s \geq s_2(t)$*, then* $\sigma_s(\mathrm{Split}_3(\mathbb{P}^n)$ *has the expected dimension.*

Proof. The proof consists of two parts. The first part is to check the abundancy of the statement $\mathfrak{B}_i(19 - 6 \cdot (3-i) - j, 3, s_k)$ for $i \in \{0, \dots, 3\}$, $j \in \{1, \dots, 6\}$, and $k \in \{1, 2\}$. This is equivalent to showing $b_{i,s_k}(19 - 6 \cdot (3-i), 3) \leq \binom{19-6\cdot(3-i)+3}{3}$ for such i, j, and k, which can be done exactly in the same way as in the proof of Theorem 4.2, because a_{i,s_k} and b_{i,s_k} are the same polynomial for each $k \in \{1, 2\}$.

The second part is to show the truth of the statements $\mathfrak{B}_3(19, 3, s_k)$ and $\mathfrak{B}_i(19 - 6 \cdot (3-i) - j, 3, s_k)$ for all $i \in \{0, \dots, d-1\}$ and for all $j \in \{1, \dots, \ell\}$. To do so, all we have to do is define the ideal generated by (5.1) and check that it is minimally generated by the suggested number of cubic forms. This can be done with the aid of a computer algebra. □

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