Problem 1 (2 points)
Assume that $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{Z}$. Use this assumption and induction to prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for all integers $n \geq 2$ and arbitrary integers $a_1, a_2, \ldots, a_n$.

Problem 2 (2 points)
Show that if the statement

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n$$

is assumed to be true for some $n$, then it can be proved to be true for $n + 1$. Is the statement true for all $n \in \mathbb{N}$?

Problem 3 (2 points)
Let $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$. Define a binary operation on $\mathbb{Z}_n$ by

$$[a][b] = [ab]$$

for all $[a], [b] \in \mathbb{Z}_n$. Prove that it is well-defined, i.e., prove that if $[a] = [a']$ and $[b] = [b']$, then $[a][b] = [a'][b']$.

Problem 4 (2 points)
In each of the following, a rule is given that determines a binary operation $*$ on $\mathbb{Z}$. Determine in each case whether $*$ is commutative or associative and whether there is an identity element.

(i) $x * y = x + y + 3$.

(ii) $x * y = x + xy$. 
Problem 5 (2 points)
Let $S$ be a set of three elements given by $S = \{A, B, C\}$. In the following table, all the elements of $S$ are listed in a row at the top and in a column at the left. The result of $x \ast y$ is found in the row that starts with $x$ at the left and in the column that has $y$ at the top. For example, $B \ast C = C$ and $C \ast B = A$.

\[
\begin{array}{ccc}
& A & B & C \\
A & C & A & B \\
B & A & B & C \\
C & B & A & C \\
\end{array}
\]

(i) Is the binary operation $\ast$ commutative? Why?
(ii) Determine whether there is an identity element in $S$ with respect to $\ast$.
(iii) If there is an identity element, which elements have inverses?