Problem 1 (4 points)
Assume that
\[ R = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \mid x, y \in \mathbb{Z} \right\} \]
is a ring with respect to matrix addition and multiplication.

(i) Let \( \varphi : R \to \mathbb{Z} \) be the map defined by
\[ \varphi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = x. \]
Prove that \( \varphi \) is an onto ring homomorphism.

(ii) Describe \( \ker(\varphi) \) and exhibit an isomorphism from \( R/\ker(\varphi) \) to \( \mathbb{Z} \).

Problem 2 (2.5 points)
Let \( I \) and \( J \) be ideals of a ring \( R \). Prove that \( I/(I \cap J) \) is isomorphic to \( (I + J)/J \).

Hint. Define a map \( \varphi : I \to (I + J)/J \) by \( \phi(x) = x + J \). Prove that \( \varphi \) is an onto ring homomorphism and that \( \ker(\varphi) = I \cap J \). Then use the first isomorphism theorem to prove that \( I/(I \cap J) \) is isomorphic to \( (I + J)/J \).

Problem 3 (2.5 points)
Assume that
\[ R = \left\{ \begin{pmatrix} m & 2n \\ n & m \end{pmatrix} \mid m, n \in \mathbb{Z} \right\} \]
and
\[ R' = \left\{ m + n\sqrt{2} \mid m, n \in \mathbb{Z} \right\} \]
are rings with respect to their usual operations. Prove that \( R \) and \( R' \) are isomorphic.

Problem 4 (2.5 points)
Let \( R \) and \( R' \) be rings with unities \( 1_R \) and \( 1_{R'} \) respectively. Prove that if \( \phi : R \to R' \) is an onto ring homomorphism, then \( \phi(1_R) = 1_{R'} \).