Problem 1 (Mixed)
Make each of the following true or false.

(i) Every field is a UFD.
(ii) Every UFD is a PID.
(iii) If $D$ is a UFD, then $D[x]$ is a UFD.
(iv) $\mathbb{C}$ is a simple extension of $\mathbb{R}$.
(v) $\mathbb{Q}$ is an extension of $\mathbb{Z}_2$.
(vi) Every non-constant polynomial in $F[x]$ has a zero in some extension field of $F$.
(vii) Every finite extension of a field is an algebraic extension.
(viii) Every algebraic extension of a field is a finite extension.

Problem 2 (UFD)
Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

Problem 3 (Prime elements and irreducible elements)
Prove that if $p$ is irreducible in a UFD, then $p$ is a prime.

Problem 4 (Fields)
Let $F$ and $F'$ be fields and let $\varphi : F \to F'$ be a ring homomorphism. Prove that either $\varphi$ is the zero map or $\varphi$ is one-to-one.

Problem 5 (Field extensions)
Let $f(x) = x^3 + x + 1 \in \mathbb{Q}[x]$.

(i) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
(ii) Let $\alpha$ be a zero of $f(x)$ in $\mathbb{C}$. Find $\alpha^{-1}$ and $(\alpha^2 + \alpha + 1)^{-1}$ in $\mathbb{Q}(\alpha)$. 
Problem 6 (Field extensions)
Let $E$ be an extension of a field $F$. Suppose that $E_1$ and $E_2$ are subfields of $E$ containing $F$. Prove that if $[E_1 : F]$ and $[E_2 : F]$ are primes and if $E_1 \neq E_2$, then $E_1 \cap E_2 = F$.

Problem 7 (Algebraic elements)
In (i) and (ii), show that the given number $\alpha$ is algebraic over $\mathbb{Q}$ by finding $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$.

(i) $\alpha = 1 + i$.

(ii) $\alpha = \sqrt{1 + \sqrt{2}}$.

Problem 8 (Minimal polynomials)
Find $[\mathbb{Q}(\sqrt{2} + i) : \mathbb{Q}]$.

Problem 9 (Algebraic extensions)
Let $F$ be an extension of a field with $q$ elements and let $E$ be an extension of $F$. Suppose that $\alpha \in E$ is algebraic over $F$. Prove that $|F(\alpha)| = q^n$ for some positive integer $n$.

Problem 10 (Simple extensions)
Prove that $\mathbb{Q}(\sqrt{3} + \sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

Problem 11 (Splitting fields)
Find the splitting field for $x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$.

Problem 12 (Splitting fields)
Find the splitting field for $x^4 - x^2 - 2 \in \mathbb{Z}_3[x]$. 