Ch. 3 Linear Equations of Higher Order (Cont'd)

Ex. (a) \( x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0 \)

(i) We showed last time that \( x^2 \) is a solution

(ii) \( C_1 x^2 \) is a solution

\[
x^2 \cdot 2C_1 - 2x \cdot 2C_1 x + 2 \cdot C_1 x^2 \overset{?}{=} 0 \quad o = 0 \checkmark
\]

(iii) \( x \) is a solution

\[
x^2 \cdot 0 - 2x \cdot 1 + 2 \cdot x = 0 \checkmark
\]

(iv) \( C_2 x \) is a solution

\[
x^2 \cdot 0 - 2x \cdot C_2 + 2 \cdot C_2 x = 0 \checkmark
\]

(v) \( C_1 x^2 + C_2 x \) is also a solution

\[
x^2 \cdot 2C_1 - 2x \left( 2C_1 x + C_2 \right) + 2 \left( C_1 x^2 + C_2 x \right) \overset{?}{=} 0 \quad C_1 \cdot 0 + C_2 \cdot 0 = 0
\]
(8) \[ \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = 0 \] is a 2nd order DE, homogeneous w/ constant coefficients.

(i) \( e^{-x} \) is a solution.

\[ e^{-x} + 3(-e^{-x}) + 2e^{-x} = 0 \] \( \checkmark \)

(ii) \( C_1 e^{-x} \) is a solution.

(iii) \( e^{-2x} \) is a solution.

(iv) \( C_2 e^{-2x} \) is a solution.

(v) \( C_1 e^{-x} + C_2 e^{-2x} \) is a solution.

Thm  Principle of Linear Superposition

Given \( a_2(x) y'' + a_1(x) y' + a_0(x) y = 0 \) is a 2nd order, linear homogeneous DE.

If \( y_1(x) \) and \( y_2(x) \) are solutions of this DE, then their linear combination
$C_1 y_1(x) + C_2 y_2(x)$
is also a solution of this DE, where $C_1$ and $C_2$ are arbitrary constants.

In general, for $n^{th}$ order linear homogeneous DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y = 0$$

if $y_1(x), y_2(x), \ldots, y_n(x)$ are solutions of this DE, then their linear combination

$$C_1 y_1(x) + C_2 y_2(x) + \ldots + C_n y_n(x)$$
is also a solution of this DE, where $C_1, C_2, \ldots, C_n$ are arbitrary constants.
Operator notation for linear DEs with constant coefficients

Denote by \( D = \frac{d}{dx} \), then

\[
\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = DD = D^2
\]

Then we can write a DE w/ constant coefficients

\[
an_d\frac{d^ny}{dx^n} + an_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0
\]

as

\[
an_D^n y + an_{n-1}D^{n-1}y + \ldots + a_2D^2y + a_1Dy + a_0y = 0
\]

or

\[
\left( anD^n + an_{n-1}D^{n-1} + \ldots + a_2D^2 + a_1D + a_0 \right)y = 0
\]

\[P(D)\]
\[ P(D) \text{ is a differential operator} \]

\[ P(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_2 D^2 + a_1 D + a_0 \]

is a polynomial in \( D \) of degree \( n \).

\[ \text{Note every linear homogeneous DE w/ constant coefficients can be written as } P(D) y = 0 \text{ and vice versa, for every operator polynomial } P(D) \text{, there is a DE } P(D) y = 0: \text{ linear DE, homogeneous, w/ constant coefficients.} \]

Ex. Write the following DEs using operator notation.

(a) \[ \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + 13 y = 0 \]

\[ (D^2 + 6D + 13) y = 0 \]

(b) \[ y'' + y'' - 6y y'' + 17y' - 6y = 0 \]

\[ (D^4 + D^3 - 6y D^2 + 17D - 6) y = 0 \]
\[
(D-a)(D-b)y = (D-b)(D-a)y = \left[D^2-(a+b)D+ab\right]y
\]

ie.
\[
\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right)y = \left(\frac{d}{dx}-b\right)\left(\frac{d}{dx}-a\right)y = \left(\frac{d^2}{dx^2}-(a+b)\frac{d}{dx}+ab\right)y
\]

\[
\left(\frac{d}{dx}-a\right)\left(\frac{d}{dx}-b\right)y = \left(\frac{d}{dx}-a\right)\left(\frac{dy}{dx} - by\right) = \frac{d^2y}{dx^2} - b \frac{dy}{dx} - a \left(\frac{dy}{dx} - by\right)
\]

\[
= \frac{d^2y}{dx^2} - (a+b) \frac{dy}{dx} + ab \cdot y = \left[D^2-(a+b)D+ab\right]y
\]

*Note: This shows a property of the operator \(PD\) which it shares with polynomials, i.e. \(PD\) can be written in a form as if "polynomial" \(PD\) were factored out. It is true for polynomials \(PD\) of any degree.*
The equation given is:

\[ \frac{d^3y}{dx^3} - 6 \frac{d^2y}{dx^2} + 11 \frac{dy}{dx} - 6y = 0 \]

This can be rewritten as:

\[(D^3 - 6D^2 + 11D - 6)y = 0\]

Or:

\[(D-1)(D-2)(D-3)y = 0\]

Or:

\[(D-2)(D-3)(D-1)y = 0\]

\[(D-3)(D-1)(D-2)y = 0\]

Exercise: Evaluate \(D^2 + 2D + 1\) operating on \(e^x\), \(sinx + 3\), \(e^{-x}\), \(xe^{-x}\).

\((D^2 + 2D + 1)e^x = e^x + 2e^x + 1 \cdot e^x = 4e^x\)

\((D^2 + 2D + 1)(sinx + 3) = -6sinx + 2 \cdot cosx + (6sinx + 3) = 2cosx + 3\)

\((D^2 + 2D + 1)e^{-x} = e^{-x} + 2(-e^{-x}) + 1 \cdot e^{-x} = 0\)

\((D^2 + 2D + 1)(xe^{-x}) = (-e^{-x} - e^{-x} + xe^{-x}) + 2(e^{-x} - xe^{-x}) + 1 \cdot (xe^{-x}) = 0\)
In this example, we saw that operator \( D^2 + 2D + 1 \) when operating on \( e^x \) and \( xe^x \) produced 0. But this is precisely what we need in order to solve the DE

\[
(D^2 + 2D + 1) y = 0
\]  

(\#)

Hence, functions \( e^{-x} \) and \( xe^x \) are solutions of 

\[
(D^2 + 2D + 1) y = 0
\]

This means that to solve DE

\[
(D^2 + 2D + 1) y = 0
\]

we seek functions that the operator \( D^2 + 2D + 1 \) ANNIHILATES.

Note: Since \( e^{-x} \) and \( xe^x \) are solutions of \( (D^2 + 2D + 1) y = 0 \)

so \( y'' + 2y' + y = 0 \), their linear combination

\[
C_1 e^{-x} + C_2 xe^x
\]

is also a solution of DE (\#).
Ex. Solve \( y^{(3)} - 6y'' + 11y' - 6y = 0 \)

\[ (D^3 - 6D^2 + 11D - 6)y = 0 \]

Since we can permute the order of the operator \( D \), we will have three solutions.

\[ (D - 1)(D - 2)(D - 3)y = 0 \]