Complex roots with nonzero real part

Ex. Solve \( y'' + 4y' + 13y = 0 \)

\[
(D^2 + 4D + 13) y = 0
\]

\[
D^2 + 4D + 13 = (D^2 + 4D + 4) + 9 = (D + 2)^2 + 3^2
\]

Complete square

\[
(D + 2)^2 + 3^2 = 0
\]

\[
(D + 2)^2 = -3^2
\]

\[
D = -2 \pm 3i: \text{ roots}
\]

Another way:

\[
D^2 + 4D + 13 = 0
\]

\[
\frac{-4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} \quad = \quad -4 \pm \frac{\sqrt{4 - 13}}{2} \quad = \quad -4 \pm \frac{\sqrt{-9}}{2} \quad = \quad \frac{-4 \pm 2 \cdot i \cdot 3}{2} \quad = \quad -2 \pm 3i
\]

same as above

\[
y(x) = C_1 e^{(-2 + 3i)x} + C_2 e^{(-2 - 3i)x}
\]

\[
e^{a+b} = e^a \cdot e^b
\]

\[
\text{Euler identities}
\]
\[ C_1 e^{-2x} \left( \cos 3x + i \sin 3x \right) + C_2 e^{-2x} \left( \cos 3x - i \sin 3x \right) = \]
\[ = \left( C_1 + C_2 \right) e^{-2x} \cos 3x + i \left( C_1 - C_2 \right) e^{-2x} \sin 3x = \]
\[ = k_1 e^{-2x} \cos 3x + i k_2 e^{-2x} \sin 3x \]

We can verify directly that \( e^{-2x} \cos 3x \) and \( e^{-2x} \sin 3x \) are also solutions of \( y'' + y' + 13y = 0 \) (and they are real-valued)

\[ \therefore y(x) = C_1 e^{-2x} \cos 3x + C_2 e^{-2x} \sin 3x \]

**General Solution**

**Operator Identity**

\[ \left[ (\mathcal{A} - a)^2 + b^2 \right] e^{ax} \cos bx = 0 \]

\( a = \pm i b \)
OPERATOR IDENTITY \[ V \]
\[
\left( D-a^2 + b^2 \right)^n x^k e^{ax} \cos bx \left( D-a^2 + b^2 \right)^n x^k e^{ax} \cos bx = 0 \quad k=0,1, \ldots, n-1
\]
\[ a \pm ib, \ldots, a \pm ib \]
\[ \text{n times} \]

HW on sections 3.1-3.3: due this Friday

LINEAR DEPENDENCE AND INDEPENDENCE. WRONSKIAN

Recall defs of hyperbolic cosine and hyperbolic sine functions:
\[
\cosh (ax) = \frac{e^{ax} + e^{-ax}}{2}
\]
\[
\sinh (ax) = \frac{e^{ax} - e^{-ax}}{2}
\]
\[
\frac{d}{dx} \cosh (ax) = a \sinh (ax)
\]
\[
\frac{d}{dx} \sinh (ax) = a \cosh (ax)
\]

Aside
\[
\cos ax = \frac{e^{iax} + e^{-iax}}{2}
\]
\[
\sinh ax = \frac{e^{iax} - e^{-iax}}{2i}
\]
Consider DE
\[ d^2y + g(x) = 0 \]
\[ \frac{d^2y}{dx^2} + 1 \]
Solutions: \( e^x, e^{-x}, 3e^x, e^{-x} \)

Can we write the general solution as \( y(x) = C_1 e^x + C_2 e^{-x} \)?

Yes

How about \( y(x) = C_1 e^x + C_2 e^{-x} + C_3 e^{\frac{x}{2}} \)?

No

\[ y(x) = (C_1 e^x + C_2 e^{-x}) e^{\frac{x}{2}} \]
Only two arbitrary constants, not three! We were able to reduce to two "essentially different" (linearly independent) functions.

Def: A linear combination of functions \( f_1(x), f_2(x), \ldots, f_n(x) \) is

\[ c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) \]

where \( c_1, c_2, \ldots, c_n \) are arbitrary constants.

E.g. \( \sinh x = \frac{1}{2} e^x - \frac{1}{2} e^{-x} \)

Def: A set of functions \( f_1(x), f_2(x), \ldots, f_n(x) \), defined on a common interval \( I \), is said to be linearly dependent (LD) if there exist a set of constants \( c_1, c_2, \ldots, c_n \), not all being zero, such that the linear combination

\[ c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \quad \text{for all} \quad x \in I \]
If at least one constant, say $C_i$, is non-zero, then we can write

$$f_i(x) = -\frac{C_1}{C_i} f_1(x) - \frac{C_2}{C_i} f_2(x) - \ldots - \frac{C_{i-1}}{C_i} f_{i-1}(x) - \frac{C_{i+1}}{C_i} f_{i+1}(x) - \ldots$$

and

$$\ldots - \frac{C_n}{C_i} f_n(x)$$

i.e. we expressed $f_i(x)$ as a linear combination of the rest of functions.

Is the set of functions $\{e^x, e^{-x}, \sin x, \cos x\}$ linearly dependent?

To answer this question, we have to form a linear combination of given functions and set it to zero. Then we try to find constants $C_1, C_2, C_3, C_4$, not all being zero, such that this linear combination is zero for all $x$. 
Thus,
\[ c_1 e^x + c_2 e^{-x} + c_3 \sin x + cy \cdot 4 \cos x = 0 \]
\[ c_1 e^x + c_2 e^{-x} + c_3 \sin x + cy \cdot 2 (e^x + e^{-x}) = 0 \]
\[ (c_1 + 2cy)e^x + (c_2 + 2cy)e^{-x} + c_3 \sin x = 0 \]

\[ c_1 + 2cy = 0 \]
\[ c_2 + 2cy = 0 \]
\[ c_3 = 0 \]

Let \[ c_4 = 1 \] \Rightarrow \[ c_1 = -2 \]
\[ c_2 = -2 \]

\[ c_1^2 + c_2^2 + c_3^2 + c_4^2 \neq 0 \]

Since we were able to find a set of constants \[ c_1, c_2, c_3, c_4 \] not all being zero, functions \[ e^x, e^{-x}, \sin x, 4 \cos x \] are L.D.
Note: The choice of constants is not unique.

\[ C_y = -2, \quad C_i = -y, \quad C_2 = -y, \quad C_3 = 0 \]

\[ C_y = -\frac{1}{2}, \quad C_i = 1, \quad C_2 = 1, \quad C_3 = 0 \]

Def: A set of functions is linearly independent (LI) if they are not linearly dependent.

To show that functions are linearly independent, we must show that

\[ C_1f_1(x) + C_2f_2(x) + \ldots + C_nf_n(x) = 0 \]

for all \( x \in T \)

implies \( C_1 = C_2 = \ldots = C_n = 0 \).

Q: Is the set of functions \( L_D \) or LI?

\( \forall x \in I \), \( f(x) \in \mathbb{R} \).
For all $x$

Since this is true for all $x$, it has to be true at $x = 0$

$x = 0$: $C_1 e^0 + C_2 x e^0 + C_3 x = 0$  $C_1 = 0$

$x \neq 0$: $C_2 x e^x + C_3 = 0$  $C_2 e^x + C_3 = 0$

$x = 1$: $C_2 e^1 + C_3 = 0$  $C_3 = -C_2 e^1$

$x = -1$: $C_2 e^{-1} + C_3 = 0$  $C_3 = -C_2 e^{-1}$

$\uparrow$

$C_1 = C_2 = C_3 = 0$

$\Rightarrow$ for $x \neq 0$, $x \neq 0$.

$x \in \mathbb{R}$, $x \neq 0$, $x \neq 0$.