To get graph of \( f(t-a) \), we shift graph of \( f(t) \) by \( a \) units to the right.

\[
x(t) \approx 0.69 \cos [8(t+0.03)]
\]

From the graph of \( x(t) \) we see that max displacement to the right occurs for the first time at \( t = -0.03 + \frac{\pi}{4} = 0.755 \) (s).

Note alternatively, to find the time when we have max displacement to the right we solve \( \cos [8(t+0.03)] = 1 \) for \( t \) and find the smallest \( t > 0 \).

Note if we need to find time when mass crosses equilibrium position for the first time, we would solve \( x(t) = 0 \) for smallest \( t \), i.e. \( 0.69 \cos [8(t+0.03)] = 0 \Rightarrow \cos [8(t+0.03)] = 0 \Rightarrow \text{solve for } t \).
DAMPED MOTION

Suppose that in the previous model there is an additional force, a damping, which is proportional to velocity and always opposite in sign to the velocity vector. Let \( c \) be a proportionality constant (\( c \) is a damping coefficient).

\[ c > 0 \]

\[ \begin{array}{c}
  k \\
  m \\
  \hline
  x(t)
\end{array} \]

\( x(t) \): displacement from equilibrium position

\[ m \ddot{x} = -kx - cx \]

\[ m\ddot{x} + cx + kx = 0, \quad x(0) = x_0, \quad \dot{x}(0) = v_0 \]
\[ (mD^2 + cD + k)x = 0 \]

\[ \text{roots: } \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \]

Whether the roots are real distinct, real repeated or complex conjugate depends on the sign of discriminant \( c^2 - 4mk \).

**Case 1: Overdamped Case** \( c^2 - 4mk > 0 \) or \( c^2 > 4mk \)

2 real distinct roots (negative)

\[ \frac{-c + \sqrt{c^2 - 4mk}}{2m} < 0 \]

\[ \frac{-c - \sqrt{c^2 - 4mk}}{2m} < 0 \]

\[ \equiv -p_1 \]

\[ \equiv -p_2 \]

Then

\[ x(t) = C_1 e^{-p_1 t} + C_2 e^{-p_2 t} \]

To find \( C_1, C_2 \) we use ICs:

\[ x(0) = x_0, \quad x'(0) = v_0 \]

**Note:** mass can cross equilibrium at most once
Case 2 Critically DAMPED \( c^2 = 4mk \Rightarrow 0 \) or \( c^2 = Ymk \)

2 real repeated roots \( \frac{-c}{2m} \), \( \frac{-c}{2m} \)

\[ x(t) = C_1 e^{-pt} + C_2 te^{-pt} \]

Let \( f(t) = te^{-pt} \). What is \( \lim_{t \to \infty} f(t) \) ?

\[ \lim_{t \to \infty} f(t) = \lim_{t \to \infty} te^{-pt} = 0.0 = \lim_{t \to \infty} \frac{t}{e^{pt}} = \text{rule} \quad \lim_{t \to \infty} \frac{1}{pe^{pt}} = 0 \]

Possible graphs of \( x(t) \)

Mass cannot cross equilibrium position \( x(0) = 0 \) more than once.
Case 3: Underdamped or Oscillatory Motion

\[ c^2 - 4mk < 0 \text{ or } c^2 > 4mk \]
\[ \pm \frac{c}{2} \pm \sqrt{c^2 - 4mk} \]
\[ 2m \]

Solution:
\[ x(t) = C_1 e^{-pt} \cos(\omega t) + C_2 e^{-pt} \sin(\omega t) = e^{-pt} \left( C_1 \cos(\omega t) + C_2 \sin(\omega t) \right) \]

where
\[ A = \sqrt{C_1^2 + C_2^2} \]
\[ \tan \theta = \frac{C_2}{C_1} \]

and

\[ \frac{A e^{-pt}}{A e^{-pT}} = \frac{2\pi}{T} = \frac{2\pi}{2\omega} \]

\[ T = \frac{2\pi}{\omega} \]

\[ \omega \text{: Pseudo-frequency} \]
\[ T \text{: Pseudo-period} \]
\[ A \text{: Time-varying amplitude} \]
The Simple Pendulum

Consider a weightless rod of length L. The mass m is attached to one of its ends.

We will use conservation of energy to derive DE for $\Theta(t)$.

The total energy is constant.

Total energy = Kinetic energy + Potential energy

Arc OB has length $s = L\Theta$

Then velocity of mass $m$ is $v = s' = L\dot{\Theta}$ or $v = \frac{ds}{dt} = L\frac{d\Theta}{dt}$

Kinetic energy = $KE = \frac{mv^2}{2} = \frac{m}{2} (L\frac{d\Theta}{dt})^2$

Potential energy = $PE = mgh$

From $\triangle ADB$ : $\cos \Theta = \frac{AD}{AB} \Rightarrow AD = AB \cdot \cos \Theta = L \cos \Theta$

Then $h = OD = OA - AD = L - L \cos \Theta = L(1 - \cos \Theta)$

$\therefore PE = mgh = mgL(1 - \cos \Theta)$
\[ KE + PE = \text{const} \]

\[ \frac{m}{2} L^2 \left( \frac{d\Theta}{dt} \right)^2 + m g L (1 - \cos \Theta) = \text{const} \]

\[ \frac{d^2\Theta}{dt^2} + \frac{g}{L} \sin \Theta = 0 \]

**Simple nonlinear pendulum equation**

Linearize \( \Theta \) by expanding it in Taylor series about \( \Theta = 0 \):

\[ \sin \Theta = \Theta - \frac{\Theta^3}{3!} + \frac{\Theta^5}{5!} - \ldots \]

for small \( \Theta \)

\[ \Rightarrow \sin \Theta \approx \Theta \quad \text{for small} \ \Theta \]

Then

\[ \frac{d^2\Theta}{dt^2} + \frac{g}{L} \Theta = 0 \]

**Linearized pendulum equation**

\[ \omega^2 = \frac{g}{L} \quad \omega = \sqrt{\frac{g}{L}} : \text{natural frequency} \]

\[ \Theta(t) = C_1 \cos \left( \sqrt{\frac{g}{L}} t \right) + C_2 \sin \left( \sqrt{\frac{g}{L}} t \right) = A \cos \left( \sqrt{\frac{g}{L}} t - \phi \right) \]

**Note** If \( T, L \) are known, we can compute \( g \). This is one of the ways to compute \( g \) (on Mars, for example).