Recall the problem

\[ \frac{dP}{dt} = kP \]

with solution \( P(t) = Ce^{kt} \), \( C \) is an arbitrary constant.

Condition \( P(0) = 1000 \) is called an initial condition (IC).

IC determines the unique solution out of infinitely many solutions defined by an arbitrary constant \( C \).

IC specifies \( C \)

**Def** The order of a DE is the order of the highest derivative in DE.

\[ y' = e^t : \text{1st order DE} \]

\[ \frac{d^2x}{dt^2} + g x = 0 : \text{2nd order DE} \]
$y'' + 3y^3 = 2x$ : 2nd order DE

$\frac{d^2y}{dx^2} = y \cdot y \cdot y$

$y^{(4)} + y^2 x + y = 5n x$ : $n^{th}$ order DE

$y^{IV} = \frac{d^4y}{dx^4}$

In general, $n^{th}$ order DE for $y = y(x)$ can be written as

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

A continuous function $u = u(x)$ is a solution of DE (1) on some interval $x \in I$ if

$$F(x, u, u', \ldots, u^{(n)}) \equiv 0 \quad \text{for all } x \in I$$
If we have a DE for function \( u(x) \) of one variable \( x \), then we have an ordinary differential equation (ODE).

Consider \( u = u(t,x) \): temperature of a long thin rod uniform

Then

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad k > 0: \text{thermal diffusivity}
\]

Eq. (2) is a partial differential equation (PDE). Many PDEs.

This is an ODE class.
We will start from 1st order DEs.

\[ \frac{dy}{dx} = f(x, y) \quad y = y(x) \]

\[ y(x_0) = y_0 : \quad \text{IC} \]

DE + IC form an initial value problem (IVP).

To solve IVP, we need to find a function \( y = y(x) \) that satisfies both DE and IC.

1.2 Integrals as General and Particular Solutions

Consider

\[ \frac{dy}{dx} = f(x) \]

1st order DE
We want to find solution $y(t)$.

\[ n y(x) = \int \frac{dy}{dx} \, dx = \int f(t) \, dt + C \]

A general solution of DE $\frac{dy}{dx} = f(t)$

\[ y(x) = \int f(x) \, dx + C \]

A general solution of a 1st order DE involves an arbitrary constant. It defines a one-parameter family of solutions.

Let $G(x)$ be an antiderivative of $f(t)$, i.e. $G'(x) = f(x)$.

\[ \Rightarrow y(x) = G(x) + C \]

If we have IC $y(x_0) = y_0$. 

\[ \Rightarrow \text{at } x = x_0 : \quad y(x_0) = G(x_0) + C \quad \Rightarrow \]

\[ \Rightarrow \text{evaluate} \quad C = y_0 - G(x_0) \]
The particular solution of DE $y' = f(x)$ subject to IC $y(x_0) = y_0$.

Note: The general solution describes all possible solutions of a DE.

Ex: Solve IVP

\[
\frac{dy}{dx} = 4x - 5, \quad y(1) = 2
\]

\[
y(x) = \int (4x - 5) \, dx = 2x^2 - 5x + C
\]

At $x = 1$, $y(1) = 2$

\[
y(1) = 2 \cdot 1^2 - 5 \cdot 1 + C \implies 2 = 2 - 5 + C \implies C = 5
\]

\[
\therefore y(x) = 2x^2 - 5x + 5
\]

the general solution

\[
\therefore y(x) = 2x^2 - 5x + 5
\]

the particular solution
Consider 2nd order DE

Integrate twice wrt x.

\[ \frac{dy}{dx} = \frac{d^2y}{dx^2} = g(x) \]

No y dependence

\[ \frac{dy}{dx} = \int g(x) \, dx + C_1 \]

\[ y(x) = \int \left( \int g(x) \, dx + C_1 \right) \, dx + C_2 \]

\[ y(x) = \int H(x) \, dx + C_1 x + C_2 \]

A general solution of

\[ y'' = g(x) \]

2nd order DE

\[ g(x) = \frac{d^2y}{dx^2} \]
Velocity and Acceleration

\( x(t) \): particle position

\( m \): mass of particle

\( F(t) \): force acting on the particle along its line of motion

\[ \frac{dx}{dt} = v(t) \]: velocity

\[ \frac{d^2x}{dt^2} = \frac{dv}{dt} = a(t) \]: acceleration

\[ x(t) = \int v(t) \, dt + x(t_0) \]

\( \text{Integral with upper variable limit} \)

\[ \frac{dx}{dt} \bigg|_{t=t_0} = v(t) \]

Newton's 2nd law of motion: \( ma = F \)

\[ m \frac{d^2x}{dt^2} = F \]: 2nd order DE for \( x(t) \)
\( x(0) = x_0 \): initial position \( \quad \int \quad \text{ICs} \)

\( \frac{dx}{dt}(0) = v_0 \): initial velocity

Assume for simplicity that \( F = \text{const} \Rightarrow a = \text{const} \) \( F = ma \)

\[ \frac{d^2x}{dt^2} = \frac{F}{m} \quad \text{or} \quad \frac{d^2x}{dt^2} = a \]

Integrate:

\[ \frac{dx}{dt} = \int \frac{d^2x}{dt^2} \, dt = \int a \, dt = at + C_1 \]

**IC:** \( x(0) = x_0 \) \( \Rightarrow \) at \( t = 0 \):

\[ \frac{dx}{dt}(0) = a \sqrt{0} + C_1 \Rightarrow C_1 = v_0 \]

\[ \frac{dx}{dt} = at + v_0 \quad \text{or} \quad v(t) = at + v_0 \]
Integrate again

\[ x(t) = \int (at + v_0) \, dt = \frac{at^2}{2} + v_0 t + C_2 \]

**IC:** \( x(0) = x_0 \implies x(0) = \frac{a \cdot 0^2}{2} + v_0 \cdot 0 + C_2 \implies C_2 = x_0 \)

\[ \therefore \quad x(t) = \frac{a}{2} t^2 + v_0 t + x_0 \]