\[(mD^2 + k)x = F_0 \cos \omega t, \quad A(D) = D^2 + \omega^2\]

\[\pm i\omega_0 \quad \pm i\omega\]

\[\omega_0 = \sqrt{\frac{k}{m}}: \text{natural frequency}\]

**Case I**: \(w = \omega_0\)

\[A(D) = D^2 + \omega_0^2\]

\[x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \left( K_1 \cos \omega_0 t + K_2 \sin \omega_0 t \right)\]

General solution

We can find that \(K_1 = 0, \quad K_2 = \frac{F_0}{2m\omega_0} \Rightarrow x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t\)

\[x(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty: \text{pure resonance}\]

We have reinforcement of the natural frequency \(\omega_0\) by
applying force with the same frequency.

\[ x(t) \]

\[ \frac{F_0}{2m\omega_0} t \]

\[ -\frac{F_0}{2m\omega_0} t \]

Case II \( w \neq \omega_0 \)

\[ x_q = K_1 \cos wt + K_2 \omega_n \sin wt \]

We can find that

\[ K_1 = \frac{F_0/m}{\omega_0^2 - \omega^2}, \quad K_2 = 0 \]

\[ \therefore x_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos wt \]
General solution is

\[ x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t \]

We have superposition of two oscillations with frequencies \( \omega_0 \) and \( \omega \).

**Beats**

\( \omega_0 \neq \omega \)

Let \( x(0) = x'(0) = 0 \) \( \Rightarrow C_1 = -\frac{F_0/m}{\omega_0^2 - \omega^2} \), \( C_2 = 0 \)

Then

\[ x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \left( \cos \omega_0 t - \cos \omega t \right) \]

**Trig. identity:** \( 2 \cos \alpha \cos \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \)

\( \omega - \beta = \omega_0 t \quad \omega + \beta = \omega t \)

\( \Rightarrow \alpha = \frac{1}{2} (\omega_0 + \omega) t \), \( \beta = \frac{1}{2} (\omega_0 - \omega) t \)
then
\[ x(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin \frac{1}{2} (\omega_0 - \omega) t \cdot \sin \frac{1}{2} (\omega_0 + \omega) t \]

Suppose that \( \omega \approx \omega_0 \Rightarrow \omega_0 + \omega \gg \omega - \omega_0 \)

\[ \sin \frac{1}{2} (\omega_0 + \omega) t : \text{ rapidly oscillating function} \]

\[ \sin \frac{1}{2} (\omega_0 - \omega) t : \text{ slowly oscillating function} \]

\[ x(t) = \left( \frac{2F_0/m}{\omega_0^2 - \omega^2} \right) \sin \frac{1}{2} (\omega_0 - \omega) t \cdot \sin \frac{1}{2} (\omega_0 + \omega) t \]

We can interpret \( x(t) \) as rapidly oscillating function of frequency \( \frac{1}{2} (\omega_0 + \omega) \) and slowly varying amplitude

\[ A(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin \frac{1}{2} (\omega_0 - \omega) t : \frac{\text{modulation / change}}{\text{of amplitude}} \]

This type of solution is called \text{BEATS}.\[ x(t) = A(t) \cos \omega t \]

\[ \text{This type of solution is called BEATS.} \]
3.7 Electric Circuits

Consider a wire through which charges measured in coulombs flow.

denotes an imaginary plane that "cuts" the wire (or reference plane).

Let \( Q(t) \) be charge (in coulombs). Define \( \frac{dQ}{dt} \), the rate at which charges flow past our reference plane. Current \( I(t) = \frac{dQ}{dt} \), measured in \( \text{coulombs/sec} = \text{amperes} \). Hence,

\[
I(t) = \frac{dQ}{dt} \quad \text{or} \quad Q(t) = \int_{t_0}^{t} I(t) \, dt + Q(t_0)
\]

If \( \frac{dQ}{dt} > 0 \), then charges flow from left to right.

If \( \frac{dQ}{dt} < 0 \), then charges flow from right to left.
Why do charges flow at all? We assume that there is some potential (voltage) that makes charges flow from higher potential to lower potential. We measure voltage $E(t)$ drop in volts. Assume that we have three elements in the circuit: inductor, resistor and capacitor. We would like to have relation between $I(t)$, $Q(t)$ and $E(t)$.

Resistor $\quad R \quad V_R = RI \quad \text{Ohms}$

Inductor $\quad L \quad V_L = L \frac{dI}{dt} \quad \text{Henries}$

Capacitor $\quad C \quad V_C = \frac{1}{C} Q \quad \text{Farads}$

Consider a circuit that has these three elements.

2nd Kirchhoff's Law: Algebraic sum of voltage drops due to elements in a circuit equals applied
voltage \( E(t) \).

\[
L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t)
\]

Recall \( \frac{dQ}{dt} = I(t) \), \( \frac{dI}{dt} = \frac{d^2Q}{dt^2} \)

\[
L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t),
\]

1st order DE but it is coupled since it depends on both \( Q(t) \) and \( I(t) \)

2nd order DE for \( Q(t) \)

\( Q(0) = Q_0 \): initial charge

\( I(0) = I_0 = \frac{dQ}{dt}(0) \): initial current