1.5 First order linear DEs

General 1st order linear DE is

$$\frac{dy}{dx} + a(x)y = R(x),$$

where $a(x) = \frac{a_0(x)}{a_1(x)}$, $Q(x) = \frac{R(x)}{Q_1(x)}$.

Def: A linear DE (1) is homogeneous if $R(x) \equiv 0$.

Otherwise, this DE is nonhomogeneous.
**Method of Integrating Factor**

Consider

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (2)$$

Let

$$\rho(x) = e^{\int P(x) \, dx}$$

integrating factor

Note

$$\frac{d\rho}{dx} = e^{\int P(x) \, dx} \cdot P(x)$$

Multiply both sides of (2) by

$$e^{\int P(x) \, dx}$$

$$e^{\int P(x) \, dx} \cdot \frac{dy}{dx} + e^{\int P(x) \, dx} \cdot P(x)y = e^{\int P(x) \, dx} \cdot Q(x)$$

$$\rho(x) \cdot \frac{dy}{dx} + \frac{d\rho}{dx} \cdot P(x)y = \frac{d}{dx} \left( \rho(x) \cdot Q(x) \right)$$
Note
\[ \frac{d}{dx} \left( \rho(x) y(x) \right) = \rho'(x) \cdot y + \rho(x) \cdot y' \]

Hence
\[ \frac{d}{dx} \left[ \rho(x) y(x) \right] = \rho(x) \cdot Q(x) \]

Solve for \( y(x) \). Integrate first.
\[ \rho(x) y(x) = \int \rho(x) Q(x) \, dx + C \]

Then
\[ y(x) = \frac{1}{\rho(x)} \left[ \int \rho(x) Q(x) \, dx + C \right] : \text{do not memorize} \]

But you should remember
\[ \rho(x) = e^{\int P(x) \, dx} : \text{integrating factor} \]
\[ \rho y = \int P Q \, dx + C : \text{solve for } y \]
Solve

\[ \frac{dy}{dx} + \frac{2}{x} y = x^2 \]

\[ p(x) = e^{\int P(x) \, dx} = e^{\int \frac{2}{x} \, dx} = e^{\ln(x^2)} = e^{2 \ln|x|} = x^2 \]

\[ -p \cdot y = \int p \cdot Q \, dx + C \]

\[ x^2 \cdot y(x) = \int x^2 \cdot x^2 \, dx + C \]

\[ x^2 \cdot y(x) = \int x^2 \, dx + C \]

\[ x^2 \cdot y(x) = \frac{x^5}{5} + C \]

\[ y(x) = \frac{x^3}{5} + \frac{C}{x^2} \]
Ex. Solve

\[ x \frac{dy}{dx} - 3y = x^2 e^x, \quad x > 0 \]
not in canonical form, since coefficient of \( x \) is

Divide both sides by \( x \).

\[ \frac{dy}{dx} - \frac{3}{x} y = x \quad \Rightarrow \quad Q(x) \]

\[ \int Q(x) \, dx = e^{\int -\frac{3}{x} \, dx} = -3e^{\ln(-x)} = -3 \quad \Rightarrow \quad e \]

\[ -3e^{\ln(-x)} = e \]

\[ \int P(x) \, dx = \int x \, dx = \frac{1}{2} x^2 \quad \Rightarrow \quad P(x) \]

\[ p(x) = \int P(x) \, dx = \int \frac{1}{2} x^2 \, dx = \frac{1}{6} x^3 \quad \Rightarrow \quad p(x) \]

\[ \int p(x) \, dx = \int \frac{1}{6} x^3 \, dx = \frac{1}{24} x^4 + C \quad \Rightarrow \quad \int p(x) \, dx + C \]

\[ \frac{1}{3} y = \int p(x) \, dx + C \]

\[ \frac{1}{3} y = \frac{1}{6} x^3 + C \quad \Rightarrow \quad y = \frac{1}{2} x^3 + C \]
\[ \frac{1}{x^3} y = e^x + C \quad \forall x \neq 0 \]

\[ y(x) = x^3 (e^x + C) \]

Another form of the method of integrating factor.

\[ p(x) = e^{\int_{x_0}^{x} P(t) \, dt} \]

\[ p(x) y(x) = \int_{x_0}^{x} p(t) Q(t) \, dt + p(x_0) y(x_0) \]

\[ \frac{1}{y} = y_0 \]

\[ \Rightarrow p(x) y(x) = \int_{x_0}^{x} p(t) Q(t) \, dt + y_0 \text{ : then solve for } y(x) \]

\[ y(x_0) = y_0 : \text{ IC} \]
Problem A tank initially contains 60 gallons of brine (water + salt) in which 4 pounds of salt are dissolved. Brine containing 2 lbs of salt per gallon enters the tank at the rate of 3 gallons per minute and the well stirred mixture leaves the tank at the rate of 4 gallons per minute.

(a) How much salt is in the tank at any time? 
\[ x(t) \] ?

(b) How much salt is in the tank at \( t = 5 \)?

(c) What is the salt concentration \( c(t) \) at \( t = 0, 10, 70 \)?
Mixtures

\( x(t) \): amount of salt in tank [lbs]
\( V(t) \): volume of solution [gal]
\( c(t) \): concentration of salt

\[ c(t) = \frac{x(t)}{V(t)} \quad \left[ \frac{\text{lbs}}{\text{gal}} \right] \]

Cin: concentration of solution entering the tank \( \left[ \frac{\text{lbs}}{\text{gal}} \right] \)

rin: rate at which solution enters the tank \( \left[ \frac{\text{gal}}{\text{min}} \right] \)

\( \text{Cout} = c(t) = \frac{x(t)}{V(t)} \): concentration of solution that leaves the tank \( \left[ \frac{\text{lbs}}{\text{gal}} \right] \)

rout: rate at which solution leaves the tank \( \left[ \frac{\text{gal}}{\text{min}} \right] \)
Suppose a population $P$ of rodents satisfies the differential equation $\frac{dP}{dt} = kp^2$. Initially, there are $P(0) = 2$ rodents, and their number is increasing at the rate of $\frac{dP}{dt} = 1$ rodent per month when there are $P = 10$ rodents. How long will it take for this population to grow to a hundred rodents? To a thousand? 

- $\frac{dP}{dt} = kp^2$, solution is $P(t) = \frac{1}{C - kt}$ (from #45 or use separation of variables).

$p(0) = 2 \Rightarrow at t=0 \quad P(0) = \frac{1}{C - 4.0} \Rightarrow C = \frac{1}{2}$

$\therefore P(t) = \frac{1}{\frac{1}{2} - 4t}$

$\frac{dP}{dt} = kp^2 \Rightarrow \frac{dP}{dt} \bigg|_{P=10} = k \cdot 10^2 \Rightarrow k = \frac{1}{100} \Rightarrow P(t) = \frac{1}{\frac{1}{2} - \frac{1}{100} t} = \frac{100}{50 - t} = P(t)$

$p(t) = 100 \quad \text{when} \quad \frac{100}{50 - t} = 100 \Rightarrow 50 - t = 1 \quad \text{or} \quad t = 49 \text{ (months)}$

$p(t) = 1000 \quad \text{when} \quad \frac{100}{50 - t} = 1000 \quad \text{or} \quad 50 - t = \frac{1}{10} \Rightarrow t = 49.9 \text{ (months)}$

Note: As $t \to 50$, $P(t) \to \infty$.

$p' > 0 \Rightarrow P \to \infty \text{ as } t > 0$

$p = 0 \text{ : unstable equil. solution}$

$P = 0$ : unstable equil. solution

\[
\frac{dP}{dt} = kp^2
\]
The line tangent to the graph of $y$ at $(x, y)$ passes through the point $(-y, x)$.

Tangent line $y - y_0 = k(x - x_0)$, $k$: slope

Here slope $k = \frac{dy}{dx}$, $(x_0, y_0) = (-y, x)$

$\Rightarrow y - x = \frac{dy}{dx}(x + y)$ or \[
\frac{dy}{dx} = \frac{y - x}{x + y}
\]