Lecture 36

Systems

\[
y'_1 = f_1(y_1, \ldots, y_N) \\
y'_N = f_N(y_1, \ldots, y_N)
\]

Vector form

\[
y' = f(y) \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}
\]

Special case

\[
y'_1 = ay_1 + by_2 \\
y'_2 = cy_1 + dy_2
\]

\[
y' = Ay \quad f(y) = Ay
\]
Modified Euler's method

\[ f(y) = Ay \]
\[ f(u_n) = A \cdot u_n \]

\[ u_{n+1} = u_n + h \cdot f(u_n) \]
\[ u_{n+1} = u_n + h \cdot A u_n = (I + hA) u_n \]

- \[ k_1 = A u_n \]
- \[ k_2 = A (u_n + h k_1) = A (u_n + h A u_n) = A (I + h A) u_n = \frac{A u_n}{A u_n} \]

\[ \frac{A u_n}{A u_n} = (A + h A^2) u_n \]

\[ u_{n+1} = u_n + \frac{h}{2} (k_1 + k_2) = u_n + \frac{h}{2} (A u_n + (A + h A^2) u_n) = \]
\[ (I + \frac{h}{2} A + \frac{h}{2} A + \frac{h^2}{2} A^2) u_n \]

\[ u_{n+1} = (I + h A + \frac{h^2}{2} A^2) u_n \]
Exact solution

\[ y' = A y, \quad y(0) = y_0 \]

\[ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \]

Assume that matrix \( A \) can be diagonalized with eigenvalues \( A_1, A_2 \in \mathbb{C} \) and eigenvectors \( p_1, p_2 \).

\[ A p_1 = A_1 p_1, \quad A p_2 = A_2 p_2, \quad p_1 \neq 0, \quad p_2 \neq 0 \]

If \( A_1 \neq A_2 \), then eigenvectors \( p_1 \) and \( p_2 \) are linearly independent and can form a basis in the solution space.

Let \( y(t) \) be a solution of \( y' = A y \), then

\[ y(t) = d_1(t) p_1 + d_2(t) p_2 \]

at \( t=0 \):

\[ y(0) = d_1(0) p_1 + d_2(0) p_2 = y_0 \]

This determines \( d_1(0) \) and \( d_2(0) \).
\[ y' = Ay \implies x_1' p_1 + x_2' p_2 = A (x_1 p_1 + x_2 p_2) \]

\[ x_1' p_1 + x_2' p_2 = \frac{x_1 A p_1}{A_1 p_1} + \frac{x_2 A p_2}{A_2 p_2} \]

\[ x_1' p_1 + x_2' p_2 = 2_1 x_1 p_1 + 2_2 x_2 p_2 \]

\[ p_1, p_2 \text{ are linearly independent } \implies \]

\[ x_1' = A_1 x_1, \quad x_2' = A_2 x_2 : \text{ scalar DEs for } x_1, x_2 \]

\[ y(t) = x_1(0) e^{A_1 t} p_1 + x_2(0) e^{A_2 t} p_2 \quad \text{exact solution} \]
Numerical solution

Ex. Euler's method

\[ u_{n+1} = u_n + h f(u_n) = u_n + h A u_n = (I + h A) u_n \]

\[ u_0 = y_0 = \alpha_1(0) p_1 + \alpha_2(0) p_2 \]

\[ u_1 = (I + h A) u_0 = (I + h A) \left( \alpha_1(0) p_1 + \alpha_2(0) p_2 \right) = \]

\[ = \alpha_1(0) p_1 + \alpha_2(0) p_2 + h \alpha_1(0) \alpha_p p_1 + h \alpha_2(0) \alpha_p p_2 = \]

\[ = \alpha_1(0) \left( 1 + h A_1 \right) p_1 + \alpha_2(0) \left( 1 + h A_2 \right) p_2 \]

\[ u_2 = \alpha_1(0) \left( 1 + h A_1 \right)^2 p_1 + \alpha_2(0) \left( 1 + h A_2 \right)^2 p_2 \]

\[ u_n = \alpha_1(0) \left( 1 + h A_1 \right)^n p_1 + \alpha_2(0) \left( 1 + h A_2 \right)^n p_2 \]
\[ y(t) = x_1(0) e^{a_1 t} p_1 + x_2(0) e^{a_2 t} p_2 \]  

**Exact solution**

**Note**

The exact solution is bounded for all \( t > 0 \) and all \( y_0 \)

\[ \Rightarrow \text{Re}(a_1) \leq 0 \quad \text{and} \quad \text{Re}(a_2) \leq 0 \]

Numerical solution is bounded for all \( h > 0 \) and all \( y_0 \)

\[ \Rightarrow |1 + h a_1| \leq 1 \quad \text{and} \quad |1 + h a_2| \leq 1 \]

The region in \( h a_2 \)-plane which satisfies these conditions is called the region of absolute stability for Euler's method.
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\[ |1 + h_2| \leq 1 \]
\[ |h_2 + 1| \leq 1 \]
\[ |h_2 - (-1)| \leq 1 \]

In addition to the given inequalities, the Backward Euler method is illustrated schematically.

**Backward Euler method**

\[ y' = f(y) \]

\[ \frac{u_{n+1} - u_n}{h} = f(u_{n+1}) \]

\[ u_{n+1} = u_n + h f(u_{n+1}) \]

Similarly, the Forward Euler method is shown:

**Forward Euler method**

\[ \frac{u_{n+1} - u_n}{h} = f(u_n) \]

\[ u_{n+1} = u_n + h f(u_n) \]
This method is **implicit** (Forward Euler is **explicit**)

Claim Backward Euler's method is 1st order accurate.

**Proof**

\[ u_{n+1} = u_n + h f(u_{n+1}) \]

\[ y_{n+1} = y_n + h f(y_{n+1}) + r_n = \frac{y(t_n) + y'(t_n) \cdot h + O(h^2)}{f(t_n)} \]

For exact solution \( y(t) \):

Local truncation error

Taylor

\[ y_{n+1} = y(t_{n+1}) = y(t_n + h) = y(t_n) + \frac{y'(t_n) \cdot h + O(h^2)}{f(t_n)} \]

\[ y_n + h f(y_{n+1}) + r_n = \frac{y(t_n) + y'(t_n) \cdot h + O(h^2)}{y_n} \]
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\[ f(y_{n+1}) = y'_{n+1} = y'(t_{n+1}) = y'(t_n + h) = y'(t_n) + y''(t_n) \cdot h + O(h^2) \]

\[ y_n + h \left( y'(t_n) + y''(t_n) \cdot h + O(h^2) \right) + r_n = y_n + y'(t_n) \cdot h + O(h^2) \]

\[ \Rightarrow h^2 y''(t_n) + h \cdot O(h^2) + r_n = O(h^2) \Rightarrow r_n = O(h^2) \]

Since the local truncation error is 2nd order accurate

\[ \Rightarrow \] the global error is 1st order accurate, i.e. Backward Euler's method is 1st order accurate.