Lecture 42

Linear shooting method (Cont'd)

Recast as a system of 1st order IVPs:

Let \( u_1 = v, \ u_2 = v', \ u_3 = w, \ u_4 = w' \)

\[
\begin{align*}
\frac{d}{dx} u_1 &= u_2 \\
\frac{d}{dx} u_2 &= p(x) u_2 + q(x) u_1 + r(x) \\
\frac{d}{dx} u_3 &= u_4 \\
\frac{d}{dx} u_4 &= p(x) u_4 + q(x) u_3
\end{align*}
\]

\( a \leq x \leq b \)

\( u_1(a) = \alpha, \ u_2(a) = 0, \ u_3(a) = 0, \ u_4(a) = 1 \)

\[
\Rightarrow u(x) = u_1(x) + \frac{p - u_1(b)}{u_3(b)} u_3(x)
\]

Note: We can solve the above IVP using, for example, \( O(h^4) \) Runge-Kutta method for systems.
Summary

We could use 2 methods to solve linear heat equation:

- Linear shooting method: turn BVP into 2 IVs for $v(x)$ and $w(x) \Rightarrow u(x) = v(x) + C w(x)$

- Linear finite difference: replace derivatives by finite differences, BVP $\Rightarrow$ linear algebraic system (sparse system - lots of zeros). Can solve by Gaussian elimination or iterative methods.

Nonlinear shooting method

Application: heat flow in a thin rod

$x = a$, $x = b$  

$x$: position along rod  
$u(x)$: steady-state temperature
\[ f(x, u, u') : \text{external heat source} \]

**The steady-state heat eq:**

\[ u''(x) = -f(x, u, u') \]

\[ u(a) = \alpha \]

\[ u(b) = \beta \]

**Thm**

\[ D = \mathbb{L} (x, u, u') : \quad a \leq x \leq b, \quad -\infty < u < \infty, \quad -\infty < u' < \infty \]

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\[ D = \mathbb{L} (x, u, u') : \quad a \leq x \leq b, \quad -\infty < u < \infty, \quad -\infty < u' < \infty \]

1. \( f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u'} \) are continuous on \( D \)

2. \( \frac{\partial f}{\partial u} < 0 \) on \( D \)

3. \( |\frac{\partial f}{\partial u'}| \leq M \) on \( D \)

Then the above BVP has a unique solution.
Idea: convert BVP to IVP.

IVP: \[ u''(x) = -f(x, u, u') \]
\[ u(a) = \alpha \]
\[ u'(a) = s \]
\[ s \text{ is unknown} \]

Goal: Find \( s \) such that \( u(b) = \beta \). This is called a shooting method because we are aiming at \( u \) by choosing \( s \) so that we hit a target \( (u = \beta \text{ at } x = b) \).

Choosing \( s \)

The solution depends on \( x \) and \( s \): \( u = u(x, s) \)

We want \( u(b, s) = \beta \Rightarrow u(b, s) - \beta = 0 \)

This is a root-finding problem.
Newton's method:

1. Start with initial guess $s_0$

2. $s_{k+1} = s_k - \left[ \frac{u(b, s_k) - \beta}{\frac{\partial u}{\partial s}(b, s_k)} \right]$ for $k = 0, 1, 2, ...$

$u(b, s_k)$: solve IVP with $u'(a) = s_k$ and evaluate solution at $x = b$

Question: how do we compute $\frac{\partial u}{\partial s}(b, s_k)$?

Computing $\frac{\partial u}{\partial s}$

IVP: $u''(x) = -f(x, u(x), u'(x))$

$u(a, s) = \alpha$

$u'(a, s) = s$
Take partial derivative \( w \) \( \frac{\partial u}{\partial s} \):

\[
\frac{\partial u''}{\partial s} = - \frac{\partial f}{\partial u} = - \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial s} = - \frac{\partial f}{\partial u} \cdot \frac{\partial u'}{\partial s} \quad (*)
\]

\( u(a, s) = x \Rightarrow \frac{\partial u}{\partial s} (a, s) = 0 \)

\( u'(a, s) = s \Rightarrow \frac{\partial u'}{\partial s} (a, s) = 1 \)

Let \( \frac{\partial}{\partial s} (x, s) = \frac{\partial u}{\partial s} (x, s) \)

\( \frac{\partial}{\partial x} \frac{\partial u}{\partial s} = \frac{\partial}{\partial s} \frac{\partial u}{\partial x} = \frac{\partial u'}{\partial s} \quad (***) \)

\( \frac{\partial}{\partial x} \frac{\partial u'}{\partial s} = \frac{\partial}{\partial s} \frac{\partial^2 u}{\partial x^2} = \frac{\partial u''}{\partial s} \)

\( z(x, s) \text{ must satisfy} \)

**IVP:** \( \frac{\partial^2}{\partial u} (x, u, u') \cdot z(x, s) - \frac{\partial f}{\partial u} (x, u, u') z'(x, s) \)

\( z(a, s) = 0, \quad z'(a, s) = 1 \)
Converting to a first order system:

Let \( u_1 = u \), \( u_2 = u' \), \( u_3 = z \), \( u_4 = z' \)

\[
\begin{align*}
  u_1' &= u_2 \\
  u_2' &= -f(x, u_1, u_2) \\
  u_3' &= u_4 \\
  u_4' &= -\frac{\partial f}{\partial u}(x, u_1, u_2) \cdot u_3 - \frac{\partial f}{\partial u'}(x, u_1, u_2) \cdot u_4 \\
  u_1(a) &= \alpha, \quad u_2(a) = S_k, \quad u_3(a) = 0, \quad u_4(a) = 1
\end{align*}
\] (1)

General procedure

1. Choose \( s_0 \), set \( k = 0 \).
2. Solve IVP (1) with \( u_2(a) = S_k \).
3. Compute \( S_{k+1} = S_k - \left[ \frac{u_1(b, S_k) - \beta}{u_3(b, S_k)} \right] \)

\[ k = k + 1 \]
4. Repeat step 2 and 3 until

\[ |S_k - S_{k-1}| < tol \]

Notes

1. How quickly we converge depends on initial guess so.

2. Newton's method will converge quadratically because \( z(b_1, S) = 0 \) \( b_1, S \) \( \neq 0 \). Can show using uniqueness argument.

3. At each iteration, must use a numerical method such as \( O(h^4) \) Runge-Kutta to solve the system of four 1st order DEs (IVP).