Fixed Point Iterations

\[ x = g(x) \]
\[ x_{n+1} = g(x_n) \]

\[ x_1 = g(x_0) \]
\[ x_2 = g(x_1) \]

Sequence of \( x \) converges to root \( d \).

\[ f(x) = x^2 - 3 \]
\[ g(x) = x - \frac{x^2 - 3}{2} \]

Ex:
\[ g(x) = x - \frac{x^2 - 3}{2} \]
\[ q_2(x) = \frac{3}{x} \]

\begin{tabular}{|c|c|}
\hline
\( n \) & \( X_n \) \\
\hline
0 & 1.5 \\
1 & 1.875 \\
2 & 1.617 \\
3 & 1.810 \\
4 & 1.672 \\
5 & 1.774 \\
\hline
\end{tabular}

converges

\[ x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 3 \quad \left( \frac{1}{x} \right) \]

\[ x = \frac{3}{X} \]

\[ g_2(x) \]

\begin{tabular}{|c|c|}
\hline
\( n \) & \( X_n \) \\
\hline
0 & 1.5 = \frac{3}{2} \\
1 & 2 \\
2 & 1.5 \\
3 & 2 \\
4 & 1.5 \\
\hline
\end{tabular}

diverges
What condition on \( g(x) \) guarantees that \( x_n \to a \)?

**Thm (Existence and uniqueness of a fixed point)**

A1. \( g(x) \) maps \([a, b]\) into \([a, b]\), i.e. if \( x \in [a, b] \)

   then \( g(x) \in [a, b] \)

A2. \( g(x) \) is continuous on \([a, b]\)

A3. \( |g'(x)| \leq k < 1 \) for all \( x \in [a, b] \)

Note: \( A3 \Rightarrow A2 \)

1) \( A1 \) and \( A2 \) are satisfied \( \Rightarrow g(x) \) has a
   fixed point \( a \in [a, b] \)

2) \( A1 \) and \( A3 \) are satisfied \( \Rightarrow \) fixed point \( a \) is unique.
$A_1$, $A_3$ (and $A_2$) are satisfied
$\Rightarrow g(x)$ has a unique fixed $pt$

$A_3$ fails $\Rightarrow$ 3 fixed points
$A_1$ & $A_2$ hold

$A_1$ fails $\Rightarrow$ there may or may not be a fixed $pt$

$A_2$ fails, $A_3$ fails as well
$\Rightarrow$ there is no fixed $pt$
$A_1$ holds
Proof of (1)

If \( g(a) = a \) or \( g(b) = b \), then we are done.

Otherwise, we suppose that \( a < g(x) < b \) for all \( x \in (a, b) \).

Let \( a < g(a) \) and \( g(b) < b \).

Introduce \( h(x) = x - g(x) \). \( h(x) \) is continuous since \( g(x) \) is continuous.

\[ h(a) = a - g(a) < 0, \quad h(b) = b - g(b) > 0 \]

By Intermediate Value Thm, there is \( \alpha \in (a, b) \) such that \( h(\alpha) = 0 \) \( \Rightarrow h(\alpha) = \alpha - g(\alpha) = 0 \)

\( \Rightarrow \alpha = g(\alpha) \)

\( \therefore \alpha \) is a fixed point of \( g \) in \( (a, b) \).
Recall Mean Value Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a value $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} = \frac{f(\xi) - f(a)}{\xi - a}$$

$$f(b) - f(a) = f'(\xi)(b - a)$$

This tangent line of $f(x)$ at $\xi$ has the same slope as the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Proof of (2)

Suppose that $x_1$ and $x_2$ are two fixed points of $g(x)$ in $[a, b]$.
Then
\[ |d_1 - d_2| = \frac{|g(x_1) - g(x_2)|}{g(x_1) - g(x_2)} = \frac{|g'(x)| \cdot |d_1 - d_2|}{\text{Thm}} \]

\[ = |g'(x)| \cdot |d_1 - d_2| \leq K \cdot |d_1 - d_2| \]

\[ \Rightarrow |d_1 - d_2| \leq K |d_1 - d_2| \]

\[ \frac{(1-K)}{0} |d_1 - d_2| \leq 0 \Rightarrow |d_1 - d_2| = 0 \Rightarrow d_1 = d_2 \]

\[ 0 \leq K < 1 \Rightarrow 1-K \neq 0 \quad \therefore \text{fixed pt is unique} \]

**Thm:** (Convergence of fixed-point iterations)

**A1** and **A3** hold \[ \Rightarrow \] the sequence defined by \[ x_{n+1} = g(x_n) \]
converges for any \[ x_0 \in [a, b] \].
Proof
\[ |x - x_{n+1}| = |g(x) - g(x_n)| = M |v^T \cdot g'(x)| \cdot |x - x_n| \leq K \cdot |x - x_n| \]

\[ \Rightarrow |x - x_{n+1}| \leq K \cdot |x - x_n| \]

\[ \text{error at iter. } n+1 \]
\[ \text{error at previous iter. } n \]

\[ \cdots |x - x_{n+1}| \leq K \cdot |x - x_n| \leq K^2 |x - x_{n-1}| \leq K^3 |x - x_{n-2}| \leq \cdots \]

\[ \cdots \leq K^{n+1} |x - x_0| \]

As \( n \to \infty \), \( K^{n+1} \to 0 \) since \( K < 1 \)

\[ \Rightarrow |x - x_{n+1}| \leq K^{n+1} |x - x_0| \to 0 \text{ as } n \to \infty \]

\[ \downarrow \text{fixed} \]

\[ \Rightarrow x_{n+1} \to x \text{ as } n \to \infty \]
Def: The order of convergence of a sequence

A sequence \( \{x_n\} \) is said to converge to \( x \) with order \( r \) if there exists a constant \( C \) such that

\[
|x - x_{n+1}| \leq C |x - x_n|^r
\]

\( r \): order of convergence

Note: This is equivalent to

\[
|x - x_n| \leq C |x - x_{n-1}|^r
\]

\[\text{or } \frac{|x - x_n|}{|x - x_{n-1}|^r} \leq C\]