Conditioning of roots of a polynomial

$p(x)$: polynomial

Ex: Wilkinson polynomial is

$p(x) = (x-1)(x-2) \ldots (x-20) = x^{20} - 210x^{19} + \ldots$

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$

where $a_0, a_1, \ldots, a_n$: coefficients, $a_n \neq 0$

degree polynomial $x_j, j = 1, \ldots, n$: roots of $p(x)$

Here $x_j = j$: all distinct in this case

$$\text{cond.} = \left| \frac{1}{a_i} \frac{x_j^{i-1}}{1/p'(x_j)} \right|$$

Ex: perturb coefficient

$\text{cond.} \# \text{ of root } x_j$

Write to perturbation of coefficient $a_i$

We want to compute $\text{cond.} 16$ (cond. # for root $x_{16} = 16$)
$$\text{cond}_{16} = \frac{1 - 2.10 \cdot 16^{29-1}}{|p'(16)|}$$

$$i = 19, \; j = 16$$

**HW #4, problem #1**

$$p(x) = (x-1)(x-0.99)(x-2)$$

$$x_1 = 1, \; x_2 = 0.99, \; x_3 = 2$$

Since roots $$x_1, x_2$$ are close, they may be ill-conditioned.

$$p'(x) = (x-0.99)(x-2) + (x-1)(2x-2-0.99)$$

$$p(x) = (x-1)(x-2)(x-0.99) = (x^2 - 3x + 2)(x - 0.99) =$$

$$= x^3 - 0.99x^2 - 3x^2 + 3(0.99)x + 2x - 2(0.99)$$

$$= x^3 - 3.99x^2 + [3(0.99) + 2]x - 2(0.99)$$

$$a_3 = 1 \quad a_2 = -3.99 \quad a_1 = 3(0.99 + 2), \; a_0 = -2(0.99)$$

We need to verify if cond$_1$ and/or cond$_2$ are large.

$$\text{cond}_{1} = \left| \frac{a_i x_1^{i-1}}{|p'(x_1)|} \right| \quad i = 0, \ldots, 3$$

and

$$\text{cond}_{2} = \left| \frac{a_i x_2^{i-1}}{|p'(x_2)|} \right| \quad i = 0, \ldots, 3$$

Until you get large cond.
For example, \( i = 2 \Rightarrow a_2 = -3.99 \)

\[
\cos 1 = \frac{|a_2 x_1^{2-1}|}{|p'(x_1)|} = \frac{3.99 \cdot 1}{10^{-2}} \approx 400
\]

\[
p'(x_i) = (x_i - 0.99) (x_i - 2) + (x_i - 1) (2x_i - 2.99)
\]

\[
= (1 - 0.99) (1 - 2) + (1 - 1)
\]

\[
= -0.01
\]

HW #6 problem #4

\[
A = \begin{pmatrix}
0 & 2 & -1 \\
1 & 1 & 1 \\
2 & 0 & 1
\end{pmatrix}
\]

\[
b = \begin{pmatrix}
-2 \\
1 \\
0
\end{pmatrix}
\]

We use partial pivoting.
Interchange rows 1 and 3 ⇒

\[
P_1 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 & 0 & 1 \\
1 & 1 & 1 \\
0 & 2 & -1
\end{pmatrix}
\]

\[
m_{21} = -\frac{1}{2}, \quad m_{31} = 0
\]
\[
\begin{pmatrix}
2 & 0 & 1 \\
0 & 1 & \frac{1}{2} \\
0 & 2 & -1
\end{pmatrix}
\]

Pivoting: interchange rows 2 and 3

\[
P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
+\frac{1}{2} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
m_{32} = -\frac{1}{2}
\]

Gaussian elimination:

\[
\begin{pmatrix}
2 & 0 & 1 \\
0 & 2 & -1 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{2} + \frac{1}{2} = 1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
P = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Check: \( PA = LU \)

\[
A = \begin{pmatrix}
0 & 2 & -1 \\
1 & 1 & 1 \\
2 & 0 & 1
\end{pmatrix}
\]

\[
L = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 1 \\
0 & 2 & -1
\end{pmatrix}
\]

\[
p = \begin{pmatrix}
3 \\
1 \\
2
\end{pmatrix}
\]

\[
U
\]

permutation vector
To solve \( Ax = b \), we multiply both sides by \( P \):

\[
PAX = Pb \quad \Rightarrow \quad LUx = Pb
\]

\[
pb = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}
\]

**Step 2: Solve \( Ly = Pb \)**

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}
\]

\[y_1 = 0 \quad y_2 = -2\]

\[
\begin{array}{c}
\frac{1}{2}y_1 + \frac{1}{2}y_2 + y_3 = 1 \\
-2
\end{array}
\Rightarrow y_3 = 2
\]

\[
y = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}
\]

\[
\checkmark
\]

**Step 3: Solve \( Ux = y \)**

\[
\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}
\]
\[ x_3 = 2 \]

\[ x_2 = \left( -2 + 1 \cdot x_3 \right) / 2 = 0 \]

\[ \Rightarrow x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \]

\[ 2 \cdot x_1 + 1 \cdot x_3 = 0 \Rightarrow x_1 = -1 \]

Check: \( Ax = b \)

\[ #2c \quad A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \]

Compute \( A^{-1} \) = ...

Compute \( \| A \|, \| A^{-1} \|, \text{Cond}(A) = \| A \| \| A^{-1} \| \)

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

KW #2, problem #5

\[ p(A) < 1 \Rightarrow I - A \text{ is non-singular} \]

\[ \Rightarrow \text{Assume that } I - A \text{ is singular.} \]
If $A$ has eigenvalue $2 \Rightarrow A - 2I$ is singular

$I - A$ is singular $\Rightarrow 2 = 1$ is an eigenvalue of $A$

$p(A) = \max |2| \Rightarrow p(A) > 1$  

$2$ is an eigenvalue of $A$