HW #4: please also read Sections 1.3-1.4

How many terms/iterations are needed to guarantee that the error of the approximation does not exceed a given tolerance?

We will consider this using an example of Taylor's expansion.

Let \( f(x) = e^x \). Expand it about \( x=0 \).

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}
\]

where \( \xi \) is a pt between \( x \) and \( a \).

\( a=0 \)

\[
f(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}
\]

We would like to use Taylor polynomial

\[
P_n(x) = f(0) + f'(0)x + \ldots + \frac{f^{(n)}(0)}{n!}x^n
\]

to approximate \( f(x) \).

Q How many terms one should keep to guarantee that the (absolute) error is less than 0.01 on \([1-1,2]?)?
\[ f(x) = e^x, \quad f'(x) = \ldots = f^{(n)}(x) = e^x \]

\[ e^x = e^0 + e^0 \cdot x + \frac{e^0}{2!} x^2 + \ldots + \frac{e^0}{n!} x^n + \frac{e^0}{(n+1)!} x^{n+1} \]

\[ e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{e^\xi}{(n+1)!} x^{n+1} \]

\[ \text{error} = |e^x - P_n(x)| = \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \]

where \( \xi \) is between 0 and \( x \).

We want:
\[ \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \leq 0.01 \text{ for all } x \in (-1, 2) \]

Note that:
\[ \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \leq \max_{\xi, x} \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \leq \]

\[ \leq \max_{\xi} \left| \frac{e^\xi}{(n+1)!} \right|, \max_{x} \left| x^{n+1} \right| = \frac{e^2}{(n+1)!} \cdot 2^{n+1} \]

If we find \( n \) such that \( \text{max error} \leq 0.01 \)
then this will guarantee that the actual error will be \( \leq 0.01 \) for all \( x \)

\[ \frac{e^2}{(n+1)!} \cdot 2^{n+1} \leq 0.01 \Rightarrow n \geq ? \]
\[ n = 5: \quad \frac{e^2}{6!} \cdot 2^6 = 0.6568 \pm 0.01 \]

\[ n = 7: \quad \frac{e^2}{7!} \cdot 2^8 = 0.0469 \pm 0.01 \]

\[ n = 8: \quad \frac{e^2}{8!} \cdot 2^9 = 0.0104 \pm 0.01 \]

\[ n = 9: \quad \frac{e^2}{9!} \cdot 2^{10} = 0.002085 \leq 0.01 \]

\[ n \geq 9 \]

**Root-finding Methods (Chapter 2)**

Root-finding methods are used to solve usually nonlinear equations.

Given a function \( f(x) \), \( x \) is a root of \( f(x) \) if

\[ f(x) = 0 \]

\[ \text{ex: } f(x) = x^2 - 3x + 2, \text{ roots are } x = 1, 2 \]

**Intermediate Value Theorem**

Suppose \( f(x) \) is continuous on \([a, b]\). Let \( K \) be any number between \( f(a) \) and \( f(b) \), i.e.

\[ f(a) < K < f(b) \]

or

\[ f(b) < K < f(a) \]
Then there exist a value $x \in (a, b)$ such that $f(x) = 0$.

\begin{align*}
\text{Application: } & f(x) = 0 \\
& |f(a)| < 0 < |f(b)| \\
\Rightarrow & f(a) \cdot f(b) < 0
\end{align*}

Then there exists $x \in (a, b)$ such that $f(x) = 0$.

Idea

Check the sign of $f(\frac{a+b}{2})$. Shrink the interval that contains the root.
bisection method

\[(a_0, b_0) \in (a, b), f(a) \cdot f(b) < 0\]

\[a_0 = a, \quad b_0 = b\]

\[n = 0\]

\[x_n = \frac{a_n + b_n}{2}\]

if \(f(a_n) \cdot f(x_n) < 0\) then

\[a_{n+1} = a_n\]

\[b_{n+1} = x_n\]

else

\[a_{n+1} = x_n, \quad b_{n+1} = b_n\]

\[n = n + 1\]
\[ f(x) = x^2 - 3, \quad x \in [1, 2] \]

\[ f(1) = -2 < 0 \quad f(2) = 170 \quad \Rightarrow \text{there exists a root } a \in (1, 2) \]

\[ a = \sqrt{3} \approx 1.73205 \]

| n | \( a_0 \) | \( b_0 \) | \( x_n \) | \( f(x_n) \) | \( |a_n - x_n| \) |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 1.5 | -0.75 | 0.2321 |
| 1 | 1.5 | 2 | 1.75 | 0.0625 | 0.0179 |
| 2 | 1.5 | 1.75 | 1.625 | -0.3594 | 0.1071 |
| 3 | 1.625 | 1.75 | 1.6875 | -0.1543 | 0.0446 |
| 4 | 1.6875 | 1.75 | 1.71875 | -0.0459 | 0.0133 |