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\[ \begin{array}{ccc}
E & E & B E_k \\
\hline
A & B & C \\
\end{array} \]

Pivoting (Cont'd)

General case

Suppose \( A \) is invertible, but some pivot \( a_{kk} = 0 \)

\[ A^{(k)} = E_{k-1} \ldots E_k E_k A = \begin{pmatrix}
A^{(k)}_{11} & \cdots & A^{(k)}_{1n} \\
\vdots & & \vdots \\
A^{(k)}_{n1} & \cdots & A^{(k)}_{nn}
\end{pmatrix} \]

\[ \det A^{(k)} = \det E_{k-1} \cdot \det E_k \cdots \det E_2 \cdot \det E_1 \cdot \det A \]

\[ = 1 \cdot 1 \cdots 1 \cdot \det A = 0 \]

\[ \Rightarrow \det A^{(k)} \neq 0 \Rightarrow A^{(k)} \text{ is also invertible} \]

If \( a_{ik} = 0, i = k+1, \ldots, n \), then one can show that the first \( k \) columns of \( A^{(k)} \) are linearly dependent which contradicts the fact that \( A^{(k)} \)

is nonsingular. Let \( i \) be the first row in which \( a_{ik} \neq 0 \). Then, switch rows \( k \) and \( i \) and proceed.

\[ A^{(k)} \rightarrow P_i A^{(k)} \]

where \( P_i \) is the permutation matrix
\[ P_k = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 1 \end{pmatrix} \]

\[ \text{det } P_k = -1 \]

**Theorem:** Let \( A \) be nonsingular. Then there exists a permutation matrix \( P \) such that \( PA = LU \) where \( L \) is unit lower triangular matrix, \( U \) is upper triangular matrix.

**Proof:** We can pivot to produce

\[ E_{n-1}P \cdots E_2P E_1P A = U = \begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}^{(n)} \end{pmatrix} \]

some \( P_i \) may be identity matrices.

In general, \( P_2E_i = E_iP_2 \), but there exists another transformation matrix \( \tilde{E}_i \) such that

\[ P_2E_i = \tilde{E}_iP_2 \]
Then

\[
\tilde{E}_{n-1} \ldots \tilde{E}_2 \tilde{E}_1 \begin{pmatrix} p_{n-1} \ldots p_2 \end{pmatrix} A = \tilde{U} \\
\begin{pmatrix} p_{n-1} \ldots p_2 p_1 \end{pmatrix} A = \begin{pmatrix} \tilde{E}_{n-1} \ldots \tilde{E}_2 \tilde{E}_1 \end{pmatrix}^{-1} \tilde{U} \\
\begin{pmatrix} p_{n-1} \ldots p_2 p_1 \end{pmatrix} A = \begin{pmatrix} E_{n-1} \ldots E_2 E_1 \end{pmatrix}^{-1} L \\
\Rightarrow PA = LU
\]

**Note**

1. $P$ contains pivoting information; in practice, it is not necessary to store matrix $P$, the information can be stored in the integer pivotal vector $\{p\}$:

\[
\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} p(1) = 3 \\ p(2) = 1 \\ p(3) = 2 \end{cases}
\]

2. Since $\det P_n = \pm 1 \Rightarrow \det A = \pm a_1^{(1)} a_2^{(2)} \ldots a_m^{(m)}$
Def: Matrix $A$ is called **strictly diagonally dominant** if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad i = 1, \ldots, n$$

Def: Matrix $A$ is called **positive definite** if $x^T A x > 0$ for all $x \neq 0$.

3. $A_k = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{nk} & \cdots & a_{nn} \end{pmatrix}$: $k \times k$ leading submatrix of $A$.

**Thm:** If matrix $A$ is **strictly diagonally dominant** or $A$ is positive definite or determinants of all leading submatrices are $\det A_k \neq 0$

all nonzeros that pivot arising in Gaussian elimination are nonzero and pivoting is not needed, i.e., $A = LU$ recommended.

**Note:** In practice, pivoting is needed when pivot is small.