**Polynomial interpolation**

We consider continuous function $f$ and assume we have $x_0, x_1, \ldots, x_n$: distinct points.

**Questions**

1. Does there exist a unique polynomial $p$ of least degree which interpolates function $f$ at given points, i.e. such that $f(x_i) = p(x_i)$, $i = 0, 1, \ldots, n$?

$$\exists p \text{ with } \deg p = 1$$

$$\text{deg } q = 2$$

**Thm (uniqueness)**

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct points and $p, q$ are polynomials of degree $\leq n$ such that $p(x_i) = q(x_i)$ for $i = 0, \ldots, n$, then $p(x) = q(x)$ for all $x$. 
Proof

Use fundamental theorem of algebra... (n°th degree polynomial has exactly n roots)
Here consider \( h(x) = p(x)q(x) \)

\[ h(x_i) = 0, \quad i = 0, 1, \ldots, n; \quad n+1 \text{ points} \]

1. What is the best way to evaluate \( p(x) \) at \( x = x_i \)?

2. How large is the error \( |f(x) - p(x)| \) at \( x = x_i \)?

Def. Set \( P_n \) the set of polynomials of
degree \( \leq n \)

\[ P_n = \{ a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \mid a_i \in \mathbb{R} \} \]

Note

\( P_n \) is a vector space over \( \mathbb{R} \)

since if \( p, q \in P_n \Rightarrow p+q \in P_n \)

\[ d \in \mathbb{R}, \quad dp \in P_n \]

\[ \dim P_n = n+1 \]

The standard basis for \( P_n \) is \( \{1, x, x^2, \ldots, x^n\} \)
\[ p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \] in standard basis

\[ \ell_k(x) = \prod_{i=0, i \neq k}^n \frac{x-x_i}{x_k-x_i} \]

**Example**

\( n = 2 \), \( x_0 = 1 \), \( x_1 = 2 \), \( x_2 = 3 \)

\[ \ell_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2 - 5x + 6}{2} \]

\[ = \frac{1}{2} x^2 - \frac{5}{2} x + 3 \]

\[ \ell_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{x^2 - xy + 3}{-1} \]

\[ = x^2 + yx - 3 \]

\[ \ell_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2 - 3x + 2}{2} \]

\[ = \frac{1}{2} x^2 - \frac{3}{2} x + 1 \]

**Note**

1. \( \deg \ell_k(x) = n \)
2. \( \ell_k(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \)
Given \( f \): continuous, \( x_0, x_1, \ldots, x_n \) \( n+1 \) distinct points, define

\[
p_n(x) = \sum_{k=0}^{n} f(x_k) \cdot l_k(x) = \sum_{k=0}^{n} \frac{f(x_k)}{x-x_k} \cdot (x-a)_k(x)
\]

Interpolating polynomial

Lagrange form

Note

1. \( \text{deg } p_n \leq n \)

2. \( p_n(x_i) = \sum_{k=0}^{n} f(x_k) \cdot l_k(x_i) = f(x_i), \quad i = 0, 1, \ldots, n \)

Thus, \( p_n(x) \) interpolates \( f(x) \) at \( x_0, x_1, \ldots, x_n \) and has degree \( \leq n \).

\( \text{Ex} \) \( f(x) = \frac{1}{x} \), \( x_0 = 1 \), \( x_1 = 2 \), \( x_2 = 3 \), \( n = 2 \)

\[
p_2(x) = f(x_0) l_0(x) + f(x_1) l_1(x) + f(x_2) l_2(x)
\]

\[
= 1 \cdot \left( \frac{1}{2} x^2 - \frac{5}{2} x + 3 \right) + \frac{1}{2} (-x^2 + 4x - 3) + \frac{4}{3} \left( \frac{1}{2} x^2 - \frac{3}{2} x + 1 \right)
\]

\[
= \frac{1}{6} x^2 - x + \frac{11}{6}
\]

\[
p_2(1) = \frac{1}{6} - 1 + \frac{11}{6} = 1 \checkmark
\]

\[
p_2(2) = \frac{1}{6} \cdot 2^2 - 2 + \frac{11}{6} = \frac{4}{6} - 2 + \frac{11}{6} = \frac{4 - 12 + 11}{6} = \frac{3}{6} = \frac{1}{2} \checkmark
\]
\[ p_3(3) = \frac{1}{3} \]

**Note**

Lagrange form of interpolating polynomial \( p_n(x) \) is good because it shows that interpolating polynomial exists, but it has computational disadvantages:

1. it is expensive to evaluate \( p_n(x) \) at \( x = x_i \);

2. if we want to include an additional point \( x_{n+1} \), we have to recompute all Lagrange polynomials \( L_i(x) \).

**Def**

\[ p_n(x) = a_0 + a_1 (x-x_0) + a_2 (x-x_0)(x-x_1) + \ldots \]

\[ \ldots + a_n (x-x_0)(x-x_1) \ldots (x-x_{n-1}) \]

This is an interpolating polynomial in Newton's form.
\[ \text{Let } n \geq 1, \quad x_0, x_1 \]

\[ p_n(x) = f(x_0) l_0(x) + f(x_1) l_1(x) = \]

\[ = f(x_0) \cdot \frac{x-x_1}{x_0-x_1} + f(x_1) \cdot \frac{x-x_0}{x_1-x_0} : \text{Lagrange form} \]

\[ = f(x_0) + \frac{f(x_1)-f(x_0)}{x_1-x_0} (x-x_0) : \text{Newton's form} \]

\[ \text{Note: \quad If } n \geq 1, \text{ we need to have an algorithm of computing coefficients } a_0, a_1, \ldots, a_n \text{ in Newton's form of interpolating polynomial } p_n. \]

\[ \text{Claim: \quad There exists a number } a_n \text{ such that} \]

\[ p_n(x) = p_{n-1}(x) + a_n (x-x_0)(x-x_1) \ldots (x-x_{n-1}) \]

\[ \Rightarrow \]

\[ p_n(x_i) = p_{n-1}(x_i) + a_n (x_i-x_0)(x_i-x_1) \ldots (x_i-x_{n-1}) \]

\[ \forall 0 \leq i \leq n-1 \Rightarrow p_n(x_i) = p_{n-1}(x_i) + a_n \cdot 0 \]

For \( i = n \):

\[ f(x_n) = p_n(x_n) = p_{n-1}(x_n) + a_n (x_n-x_0)(x_n-x_1) \ldots (x_n-x_{n-1}) \]

\[ \Rightarrow f(x_n) = 0 \]
\[ f(x_n) = p_{n-1}(x_n) + a_n \left( \frac{x_n - x_0}{(x_n - x_1)(x_n - x_2) \cdots (x_n - x_{n-1})} \right) \]

\[ a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} \]

\[ \Rightarrow \] gives equation to find \( a_n \)