Piecewise polynomial interpolation

Given function \( f \), \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \)

The interpolating polynomial of degree \( n \), \( p_n \), may not give a good approximation over the entire interval \([a, b] \).

Define \( g_i \) : piecewise linear interpolant by

\[
g_i(x) = f[x_i:J] + \frac{f[x_i, x_{i+1}]}{(x-x_i)} (x-x_i), \quad x \in [x_i, x_{i+1}]
\]

\[
g_i(x_i) = f[x_i:J] = f(x_i)
\]

\[
g_i(x_{i+1}) = f[x_i:J] + \frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i} (x_{i+1}-x_i) = f(x_{i+1})
\]

\( g_i \) is continuous on \([a, b] \) but not differentiable on \([a, b] \) (\( g_i \) is not differentiable at \( x_i \), \( i = 1, 2, \ldots, n-1 \))
\[ M_i = \max |f''(x)|, \quad x \in [x_i, x_{i+1}] \]

For any \( x \in [a, b] \),
\[ |f(x) - \phi_i(x)| \leq \max_i \frac{M_i}{8} |x_{i+1} - x_i|^2 \]

\[ \|f - P_i\|_\infty \leq \frac{M}{8} (b-a)^2, \text{ where } M = \max (f'') \]

**Splines**

Let \( a = x_0 < x_1 < x_2 < \ldots < x_{n+1} = b \).

A spline of degree \( m \) is a function \( S(x) \)
that satisfies the following conditions:

1. For \( x \in [x_i, x_{i+1}] \), \( S(x) = S_i(x) \) a polynomial of degree \( \leq m \)

\[ S_i \quad S_i(x) \]
\[ x_0 \quad x_1 \quad x_i \quad x_{i+1} \quad \ldots \]

\[ \lim_{x \to x_i^-} S_i(x) = \lim_{x \to x_i^+} S_{i+1}(x) \]

\[ \lim_{x \to x_i^-} S^{(m-1)}(x) = \lim_{x \to x_i^+} S^{(m-1)}(x) \]

\[ a \text{'s'.]
\[ \text{Ex} \quad x_0 = -1, \quad x_1 = 0, \quad x_2 = 1 \]

\[ S(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ x^2, & 0 \leq x \leq 1 \end{cases} \]

Yes, \( S(x) \) is a spline of degree \( 2 = m \) (quadratic spline).

\[ \text{Ex} \quad S(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1 - (x-1)^2, & 0 \leq x \leq 1 \end{cases} \]

\[ \lim_{x \to 0^+} S'(x) = 2 \neq \lim_{x \to 0^-} S'(x) = 0 \]

No, \( S(x) \) is \textit{not} a spline since \( \lim_{x \to 0^-} S'(x) ≠ \lim_{x \to 0^+} S'(x) \)

\underline{Cubic spline interpolation}

Given \( f \), \( x_0, x_1, \ldots, x_n \) as above, find a cubic spline \( S(x) \) that interpolates function \( f(x) \): 

\[ f(x_i) = S(x_i), \quad i = 0, 1, \ldots, n \]

\( n+1 \) points \( \Rightarrow n \) intervals \( \Rightarrow 4n \) coefficients
$2n = 2(n-1) + 2$ conditions to interpolate $f$

$2(n-1)$ conditions to require that $S(x)$ and $S''(x)$ are continuous at interior points $x_1, \ldots, x_{n-1}$

$\Rightarrow$ we have $4n - 2$ conditions and $4n$ unknowns

$\Rightarrow$ we need two extra conditions

A popular choice is

$S''(x_0) = S''(x_n) = 0$: natural cubic spline

Another choice:

$S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n)$: clamped cubic spline