Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Summer 2019

Homework 2
Due: Monday, June 24, 2019

Taylor Polynomials
1. Consider the function \( f(x) = \sin(\pi x/2) \).
   (a) Expand \( f(x) \) in a Taylor series about the point \( x_0 = 0 \).
   (b) Find an expression for the remainder.
   (c) Estimate the number of terms that would be required to guarantee accuracy for \( f(x) \) within \( 10^{-5} \) for all \( x \) in the interval \([-1, 1]\).
   (d) Plot \( f(x) \) and its 1st, 3rd, 5th and 7th degree Taylor polynomials over \([-2, 2]\). (Use the Matlab command subplot to generate a number of plots on the same page).

Root Finding Methods
2. Which of the following iterations \( x_{n+1} = g(x_n) \) will converge to the indicated fixed point \( \alpha \) (provided \( x_0 \) is sufficiently close to \( \alpha \))? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence (i.e., the asymptotic constant). In the case that \( g'(\alpha) = 0 \), try expanding \( g(x) \) in a Taylor polynomial about \( x = \alpha \) to determine the order of convergence. (See Section 2.3 (pg. 90-91) for more details on convergence of fixed point iteration schemes.)
   (a) \( x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, \quad \alpha = 2 \)
   (b) \( x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \quad \alpha = \frac{3\sqrt{3}}{3} \)
   (c) \( x_{n+1} = \frac{12}{1 + x_n}, \quad \alpha = 3 \)

3. Let \( \alpha \) be a fixed point of \( g(x) \). Consider the fixed-point iteration \( x_{n+1} = g(x_n) \) and suppose that \( \max |g'(x)| = k < 1 \). Prove the following error estimate
   \[ |\alpha - x_{n+1}| \leq \frac{k}{1-k} |x_{n+1} - x_n| \]
   (hint: by MVT, \( |\alpha - x_{n+1}| = |g'(\xi)||\alpha - x_n| \leq k|\alpha - x_n| \))

4. The function \( f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7 \) has a zero at \( x = \frac{1}{3} \). Perform ten iterations of Newton’s method on this function, starting from \( p_0 = 0 \). What is the apparent order of convergence of the sequence of approximations? What is the multiplicity of the zero at \( x = \frac{1}{3} \)? Would the sequence generated by the bisection method converge faster?
5. Newton’s method approximates the zero of \( f(x) = x^3 + 2x^2 - 3x - 1 \) on the interval \((-3, -2)\) to within \(9.436 \times 10^{-11}\) in 3 iterations and 6 function evaluations. How many iterations and how many function evaluations are needed by the secant method to approximate this zero to a similar accuracy? Take \( p_0 = -2 \) and \( p_1 = -3 \).

6. (Gaussian Elimination)
   Let \( A \) be the \(2 \times 2\) matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Use Gaussian elimination to obtain \( A^{-1} \) by solving the two systems \( Ax_1 = e_1 \) and \( Ax_2 = e_2 \), where \( e_1 \) and \( e_2 \) are the columns of the \(2 \times 2\) identity matrix. Note that you can perform both at the same time by considering the augmented system \([A|I]\). Show that \( A^{-1} \) exists if and only if \( \det(A) \neq 0 \).

7. (LU Decomposition)
   Find the \(LU\) decomposition of \( A \) and use it to solve \( Ax = b \).

   \[
   A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -3 \\ 3 \\ -2 \end{pmatrix}.
   \]

8. (Back and Forward Substitution: Matlab Program)
   Write two programs, one that performs back substitution on an upper triangular matrix and another that performs forward substitution on a lower triangular matrix (you may assume that the diagonal entries are all 1). Both files should begin:

   ```
   function [x] = forwardsub(L, b)
   n=length(b);
   (your code here)
   ```

   In the above, \( Lx = b \) and \( A \) is lower triangular. Test your code on the following systems:

   \[
   \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} x = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix} y = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
   \]

   Remember, in Matlab you can solve matrix equations as follows (assuming you have defined the matrix \( A \) and the rhs vector \( b \)):

   \[
   >> A\backslash b
   \]

   Print and hand-in the text file containing your program.
9. (Special Matrices)
Consider the problem $Ax = b$ where $A$ is a tridiagonal matrix. What is the operation count for the forward elimination and the back substitution steps of Gaussian elimination in this case? Count add/sub and mult/div operations separately, then give the overall order of the total operations needed. (Use $O(n^p)$ notation).

Suggested / Additional problems for Math 529/Phys 528 students:

10. Let $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{2,1} & 1 & 0 \\ -m_{3,1} & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{3,2} & 1 \end{pmatrix}$, $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Show that $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & 0 & 1 \end{pmatrix}$.

(b) Show that $E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ m_{2,1} & 1 & 0 \\ m_{3,1} & m_{3,2} & 1 \end{pmatrix}$.

(c) Show that $P_1^{-1}=P_1$.

11. (Accelerating convergence of Newton’s method) (Please read Section 2.6 on Accelerating Convergence, in particular, on Restoring Quadratic Convergence to Newton’s method (pages 120–122))

The function $f(x) = 27x^4 + 162x^3 - 180x^2 + 62x - 7$ has a zero of multiplicity 3 at $x = 1/3$. Apply both techniques for restoring quadratic convergence to Newton’s method, discussed on pages 120–122, to this problem. Use $p_0 = 0$, and verify that both resulting frequencies converge quadratically.

12. Elementary Matrices, from Trefethen–Bau 1997

Let $B$ be a $4 \times 4$ matrix to which we apply the following operations.

- Double column 1,
- halve row 3,
- add row 3 to row 1,
- interchange columns 1 and 4,
- subtract row 2 from each of the other rows,
- replace column 4 by column 3,
- delete column 1 (so that the column dimension is reduced by 1).
(a) Write the result as a product of eight matrices, including $B$.

(b) Write it again as a product $ABC$ (same $B$) of three matrices.

Note: You may find useful using a handout on *elementary matrices* posted on the course web site:

http://www.webpages.uidaho.edu/~barannyk/Teaching/elem_matr.pdf