Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Fall 2020

Homework 3
Due: **Monday, October 12, 2020**

REVIEW sections 3.1-3.7, READ sections 3.8, 3.10.

1. **(Vector and Matrix Norms)**

   Show that the l_1 vector norm satisfies the three properties

   (a) $||x||_1 \geq 0$ for $x \in \mathbb{R}^n$ and $||x||_1 = 0$ if and only if $x = 0$

   (b) $||\lambda x||_1 = |\lambda|\ ||x||_1$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$

   (c) $||x + y||_1 \leq ||x||_1 + ||y||_1$ for $x, y \in \mathbb{R}^n$

2. **(Pivoting)**

   (a) Prove that the matrix
   $$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
   does not have an LU decomposition. Hint: assume that such decomposition exists and then show that this brings a contradiction.

   (b) Does the system
   $$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$
   have a unique solution for all $a, b \in \mathbb{R}$? (Why?)

   (c) How can you modify the system in part (b) so that LU decomposition applies?

3. **(Partial Pivoting)**

   Consider the linear system, $Ax = b$, where A is the following matrix,
   $$A = \begin{pmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{pmatrix}.$$

   (a) Using **partial pivoting technique**, determine the P, L, U decomposition of the matrix A, such that $PA = LU$. (Show **EACH STEP** in the decomposition.)

   (b) Use the P, L, U decomposition found in (a) to find the solution to $Ax = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ (Show **ALL** relevant steps).

(c) Use the P, L, U decomposition found in (a) to find the solution to $Ax = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ (Show **ALL** relevant steps).

4. **(Partial Pivoting: MATLAB program)**

   Write a program to find the LU decomposition of a given $n \times n$ matrix A using **partial pivoting**. The program should return the updated matrix A and the pivot vector p. In MATLAB, name your file mylu.m, the first few lines of which should be as follows:

```
function [a,p]=mylu(a)
%
[n n]=size(a); p=(1:n)'; (your code here!)
```

   The code above sets n equal to the dimension of the matrix and initializes the pivot vector p. Make sure to store the multipliers m_{ij} in the proper matrix entries. For more help on function m-files see pages $9-13$ of the MATLAB Primer by Kermit Sigmon available from the course webpage. You should experiment with a few small matrices to make sure your code is correct. Check if matrices resulting in LU decomposition satisfy $PA = LU$. As a test of your code, in MATLAB execute the statements

```
>>diary mylu.txt
>>format short e
>>type mylu.m
>>a=[2 2 -3;3 1 -2;6 8 1];
>>[a,p]=mylu(a)
>>diary off
```

   Print and hand-in the text file containing your program.

5. (a) Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$

   Compute $\|A\|_\infty$ and find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$.

   (b) Find an example of a 2×2 matrix A such that $\|A\|_\infty = 1$ but $\rho(A) = 0$. This shows that the spectral radius $\rho(A) = \{\max |\lambda| : \ \lambda \text{ is an eigenvalue of } A\}$ **does not** define a matrix norm.

6. Consider the matrix, right side vector, and two approximate solutions

$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \; b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \; x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}.$$

(a) Show that $x = (2, -2)^T$ is the exact solution of $Ax = b$.

(b) Compute the error and residual vectors for x_1 and x_2.

(c) Find $||A||_\infty, ||A^{-1}||_\infty$ and $\text{cond}_\infty(A)$ (you may use MATLAB for this calculation).

(d) In class we proved a theorem relating the condition number of A, the relative error, and the relative residual. Check this result for the two approximate solutions x_1 and x_2.

7. **(LU factorization)**

(a) Write a program that takes the output A and p from problem # 4, along with a righthand side b, and computes the solution of $Ax = b$ by performing the forward and backward substitution steps. If you are using MATLAB, name your m-file lusolve.m. The first line of your code lusolve.m should be as follows:

```
function x=lusolve(a,p,b)
(your code here!)
```

Turn in a copy of your code.

(b) The famous Hilbert matrices are given by $H_{ij} = 1/(i + j - 1)$. The $n \times n$ Hilbert matrix H_n is easily produced in MATLAB using *hilb(n)*. Assume the true solution of $H_n x = b$ for a given n is $x = [1, \ldots, 1]^T$. Hence the righthand side b is simply the row sums of H_n, and b is easily computed in MATLAB using *b=sum(hilb(n)')'*. Use your codes mylu.m and lusolve.m to solve the system $H_n x = b$ for $n = 5, 10, 15, 20$. For each n, using the $\infty-$norm, compute the relative error and the relative residual. Discuss what is happening here. You may find it useful to look at the *cond* command in MATLAB.

8. **(Iterative Methods: Analysis).**

Recall that an $n \times n$ matrix A is said to be *strictly diagonally dominant* if

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| < |a_{ii}| \qquad \text{for } i = 1, \ldots, n\,.$$

Note that the strict inequality implies that each diagonal entry a_{ii} is non-zero. Suppose that A is strictly diagonally dominant.

(a) Show that the Jacobi iteration matrix satisfies $||B_J||_\infty < 1$ and, therefore, Jacobi iteration converges in this case.

(b) For a 2×2 matrix A, show that the Gauss-Seidel iteration matrix also satisfies $||B_{GS}||_\infty < 1$ and, hence, Gauss-Seidel iteration converges as well.

9. **(from Mathews–Fink 2004)**

   The Rockmore Corp. is the considering the purchase of a new computer and will choose either the DoGood 174 or the MightDo 11. They test both computers' ability to solve the linear system

$$\begin{aligned} 34x + 55y - 21 &= 0 \\ 55x + 89y - 34 &= 0 \end{aligned}$$

   The DoGood 174 computer gives $x = -0.11$ and $y = 0.45$, and its check for accuracy is found by substitution:

$$\begin{aligned} 34(-0.11) + 55(0.45) - 21 &= 0.01 \\ 55(-0.11) + 89(0.45) - 34 &= 0.00 \end{aligned}$$

   The MightDo 11 computer gives $x = -0.99$ and $y = 1.01$, and its check for accuracy is found by substitution:

$$\begin{aligned} 34(-0.99) + 55(1.01) - 21 &= 0.89 \\ 55(-0.99) + 89(1.01) - 34 &= 1.44 \end{aligned}$$

   Which computer gave the better answer? Why?

   Suggested / Additional problems for Math 529 / Phys 528 students:

10. **(Special Matrices)**

    Consider the matrix

$$\begin{pmatrix} b & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

    (a) For what values of b will this matrix be positive definite? (Hint: theorem on page 215 on leading principal submatrices may be useful.)

    (b) For what values of b will this matrix be strictly diagonally dominant? (Recall that an $n \times n$ matrix A is said to be *strictly diagonally dominant* if

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| < |a_{ii}| \qquad \text{for } i = 1, \ldots, n\,.$$

    Note that the strict inequality implies that each diagonal entry a_{ii} is non-zero.)

11. Consider a linear system with matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

(a) Write down the iteration matrices B_J and B_{GS} for Jacobi's Method and Gauss–Seidel.

(b) Find the l_∞ norm and spectral radius of the iteration matrix for Jacobi and Gauss-Seidel. (Recall that the spectral radius of a matrix can be calculated by finding the roots of its characteristic polynomial.)

(c) Which of the two iterative methods will converge for an arbitrary starting point $x^{(0)}$? Why?

(d) Write a program to calculate and plot the spectral radius of $B_{\text{sor}}(\omega)$ for parameter ω in the range $(0, 2)$ in increments of 0.01. Provide the code and the plot. Based on inspection of the graph, what value of ω will lead to the fastest convergence?

(e) Use the theorem on page 234 of Bradie to calculate analytically the optimal relaxation parameter ω for SOR. Does it match the value predicted in Part (d)?

12. **Matrix Norms**

(a) Prove that if $||A|| < 1$, then

$$||(I - A)^{-1}|| \geq \frac{1}{1 + ||A||} .$$

(b) Suppose that $A \in \mathbb{R}^{n \times n}$ is invertible, B is an estimate of A^{-1}, and $AB = I + E$. Show that the relative error in B is bounded by $\|E\|$ (using an arbitrary matrix norm).

13. **(Cholesky decomposition)** (Cholesky decomposition can be used for symmetric positive definite matrices (see pages 215-217 of the textbook).))

(a) Compute the Cholesky decomposition for matrix

$$\begin{pmatrix} 16 & -28 & 0 \\ -28 & 53 & 10 \\ 0 & 10 & 29 \end{pmatrix}$$

(b) Construct an algorithm to perform forward and backward substitution on the system $Ax = b$, given a Cholesky decomposition $A = LL^T$ for the coefficient matrix. How many arithmetic operations are required by the algorithm?

(c) Solve the system $Ax = b$ with $b = \begin{pmatrix} 8 & -2 & 38 \end{pmatrix}^T$ and the above matrix A by using the Cholesky decomposition and then performing forward and backward substitution.