Math/Phys/Engr 428, Math 529/Phys 528
Numerical Methods - Spring 2020

Homework 3
Due: March 6, 2020

1. (Vector and Matrix Norms)
   Show that the \( l_1 \) vector norm satisfies the three properties
   
   (a) \( \|x\|_1 \geq 0 \) for \( x \in \mathbb{R}^n \) and \( \|x\|_1 = 0 \) if and only if \( x = 0 \)
   
   (b) \( \|\lambda x\|_1 = |\lambda| \|x\|_1 \) for \( \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}^n \)
   
   (c) \( \|x + y\|_1 \leq \|x\|_1 + \|y\|_1 \) for \( x, y \in \mathbb{R}^n \)

2. (Pivoting)
   
   (a) Prove that the matrix
   
   \[
   \begin{pmatrix}
   0 & 1 \\
   1 & 1
   \end{pmatrix}
   \]
   
   does not have an \( LU \) decomposition. Hint: assume that such decomposition exists and then show that this brings a contradiction.

   (b) Does the system
   
   \[
   \begin{pmatrix}
   0 & 1 \\
   1 & 1
   \end{pmatrix}
   \begin{pmatrix}
   x \\
   y
   \end{pmatrix} = \begin{pmatrix}
   a \\
   b
   \end{pmatrix}
   \]
   
   have a unique solution for all \( a, b \in \mathbb{R} \)? (Why?)

   (c) How can you modify the system in part (b) so that \( LU \) decomposition applies?

3. (Partial Pivoting)
   
   Consider the linear system, \( Ax = b \), where \( A \) is the following matrix,
   
   \[
   A = \begin{pmatrix}
   -5 & 2 & -1 \\
   1 & 0 & 3 \\
   3 & 1 & 6
   \end{pmatrix}
   \]
   
   (a) Using partial pivoting technique, determine the \( P, L, U \) decomposition of the matrix \( A \), such that \( PA = LU \). (Show EACH STEP in the decomposition.)

   (b) Use the \( P, L, U \) decomposition found in (a) to find the solution to
   
   \[
   Ax = \begin{pmatrix}
   2 \\
   -2 \\
   1
   \end{pmatrix}
   \] (Show ALL relevant steps).

   (c) Use the \( P, L, U \) decomposition found in (a) to find the solution to
   
   \[
   Ax = \begin{pmatrix}
   0 \\
   1 \\
   5
   \end{pmatrix}
   \] (Show ALL relevant steps).
4. (Partial Pivoting: MATLAB program)

Write a program to find the $LU$ decomposition of a given $n \times n$ matrix $A$ using **partial pivoting**. The program should return the updated matrix $A$ and the pivot vector $p$. In MATLAB, name your file mylu.m, the first few lines of which should be as follows:

```matlab
function [a,p]=mylu(a)

% [n n]=size(a); p=(1:n)'; (your code here!)
```

The code above sets $n$ equal to the dimension of the matrix and initializes the pivot vector $p$. Make sure to store the multipliers $m_{ij}$ in the proper matrix entries. For more help on function m-files see pages 9–13 of the MATLAB Primer by Kermit Sigmon available from the course webpage. You should experiment with a few small matrices to make sure your code is correct. Check if matrices resulting in $LU$ decomposition satisfy $PA = LU$. As a test of your code, in MATLAB execute the statements

```matlab
>>diary mylu.txt
>>format short e
>>type mylu.m
>>a=[2 2 -3;3 1 -2;6 8 1];
>>[a,p]=mylu(a)
>>diary off
```

Print and hand-in the text file containing your program.

5. (a) Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$  

Compute $\|A\|_\infty$ and find a vector $x$ such that $\|A\|_\infty = \|Ax\|_\infty/\|x\|_\infty$.

(b) Find an example of a $2 \times 2$ matrix $A$ such that $\|A\|_\infty = 1$ but $\rho(A) = 0$.

This shows that the spectral radius $\rho(A) = \{\max |\lambda| : \lambda \text{ is an eigenvalue of } A\}$ **does not** define a matrix norm.

6. Consider the matrix, right side vector, and two approximate solutions

$$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \quad b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}.$$  

(a) Show that $x = (2, -2)^T$ is the exact solution of $Ax = b$.

(b) Compute the error and residual vectors for $x_1$ and $x_2$.

(c) Find $\|A\|_\infty, \|A^{-1}\|_\infty$ and $\text{cond}_\infty(A)$ (you may use MATLAB for this calculation).
(d) In class we proved a theorem relating the condition number of $A$, the relative error, and the relative residual. Check this result for the two approximate solutions $x_1$ and $x_2$.

7. (LU factorization)

(a) Write a program that takes the output $A$ and $p$ from problem # 4, along with a righthand side $b$, and computes the solution of $Ax = b$ by performing the forward and backward substitution steps. If you are using MATLAB, name your m-file lusolve.m. The first line of your code lusolve.m should be as follows:

```matlab
function x=lusolve(a,p,b)
(your code here!)
```

Turn in a copy of your code.

(b) The famous Hilbert matrices are given by $H_{ij} = 1/(i + j - 1)$. The $n \times n$ Hilbert matrix $H_n$ is easily produced in MATLAB using `hilb(n)`. Assume the true solution of $H_n x = b$ for a given $n$ is $x = [1, \ldots, 1]^T$. Hence the righthand side $b$ is simply the row sums of $H_n$, and $b$ is easily computed in MATLAB using $b=sum(hilb(n)' )'$. Use your codes mylu.m and lusolve.m to solve the system $H_n x = b$ for $n = 5, 10, 15, 20$. For each $n$, using the $\infty$-norm, compute the relative error and the relative residual. Discuss what is happening here. You may find it useful to look at the `cond` command in MATLAB.

8. (Iterative Methods: Analysis).

Recall that an $n \times n$ matrix $A$ is said to be strictly diagonally dominant if

$$
\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \quad \text{for } i = 1, \ldots, n.
$$

Note that the strict inequality implies that each diagonal entry $a_{ii}$ is non-zero. Suppose that $A$ is strictly diagonally dominant.

(a) Show that the Jacobi iteration matrix satisfies $||B_J||\infty < 1$ and, therefore, Jacobi iteration converges in this case.

(b) For a $2 \times 2$ matrix $A$, show that the Gauss-Seidel iteration matrix also satisfies $||B_{GS}||\infty < 1$ and, hence, Gauss-Seidel iteration converges as well.


The Rockmore Corp. is considering the purchase of a new computer and will choose either the DoGood 174 or the MightDo 11. They test both computers’ ability to solve the linear system

$$
\begin{align*}
34x + 55y - 21 &= 0 \\
55x + 89y - 34 &= 0
\end{align*}
$$
The DoGood 174 computer gives $x = -0.11$ and $y = 0.45$, and its check for accuracy is found by substitution:

\[ 34(-0.11) + 55(0.45) - 21 = 0.01 \]
\[ 55(-0.11) + 89(0.45) - 34 = 0.00 \]

The MightDo 11 computer gives $x = -0.99$ and $y = 1.01$, and its check for accuracy is found by substitution:

\[ 34(-0.99) + 55(1.01) - 21 = 0.89 \]
\[ 55(-0.99) + 89(1.01) - 34 = 1.44 \]

Which computer gave the better answer? Why?

**Suggested / Additional problems for Math 529 / Phys 528 students:**

10. **(Special Matrices)**

Consider the matrix

\[
\begin{pmatrix}
  b & -1 & 0 \\
  -1 & 4 & 1 \\
  0 & 1 & 5
\end{pmatrix}.
\]

(a) For what values of $b$ will this matrix be positive definite? (Hint: theorem on page 215 on leading principal submatrices may be useful.)

(b) For what values of $b$ will this matrix be strictly diagonally dominant? (Recall that an $n \times n$ matrix $A$ is said to be strictly diagonally dominant if

\[ \sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}| \quad \text{for } i = 1, \ldots, n. \]

Note that the strict inequality implies that each diagonal entry $a_{ii}$ is non-zero.)

11. Consider a linear system with matrix

\[
A = \begin{pmatrix}
  2 & 1 \\
  1 & 4
\end{pmatrix}
\]

(a) Write down the iteration matrices $B_J$ and $B_{GS}$ for Jacobi’s Method and Gauss–Seidel.

(b) Find the $l_\infty$ norm and spectral radius of the iteration matrix for Jacobi and Gauss-Seidel. (Recall that the spectral radius of a matrix can be calculated by finding the roots of its characteristic polynomial.)

(c) Which of the two iterative methods will converge for an arbitrary starting point $x^{(0)}$? Why?
(d) Write a program to calculate and plot the spectral radius of $B_{\text{SOR}}(\omega)$ for parameter $\omega$ in the range $(0, 2)$ in increments of 0.01. Provide the code and the plot. Based on inspection of the graph, what value of $\omega$ will lead to the fastest convergence?

(e) Use the theorem on page 234 of Bradie to calculate analytically the optimal relaxation parameter $\omega$ for SOR. Does it match the value predicted in Part (d)?

12. Matrix Norms

(a) Prove that if $||A|| < 1$, then

\[ ||(I - A)^{-1}|| \geq \frac{1}{1 + ||A||}. \]

(b) Suppose that $A \in \mathbb{R}^{n \times n}$ is invertible, $B$ is an estimate of $A^{-1}$, and $AB = I + E$. Show that the relative error in $B$ is bounded by $\|E\|$ (using an arbitrary matrix norm).

13. (Cholesky decomposition) (Cholesky decomposition can be used for symmetric positive definite matrices (see pages 215-217 of the textbook).)

(a) Compute the Cholesky decomposition for matrix

\[
\begin{pmatrix}
16 & -28 & 0 \\
-28 & 53 & 10 \\
0 & 10 & 29
\end{pmatrix}
\]

(b) Construct an algorithm to perform forward and backward substitution on the system $Ax = b$, given a Cholesky decomposition $A = LL^T$ for the coefficient matrix. How many arithmetic operations are required by the algorithm?

(c) Solve the system $Ax = b$ with $b = (8 \ -2 \ 38)^T$ and the above matrix $A$ by using the Cholesky decomposition and then performing forward and backward substitution.