REVIEW sections 3.1-3.7, READ sections 3.8, 3.10.

1. (Vector and Matrix Norms)
   Show that the $l_1$ vector norm satisfies the three properties
   
   (a) $||x||_1 \geq 0$ for $x \in \mathbb{R}^n$ and $||x||_1 = 0$ if and only if $x = 0$
   (b) $||\lambda x||_1 = |\lambda||x||_1$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$
   (c) $||x + y||_1 \leq ||x||_1 + ||y||_1$ for $x, y \in \mathbb{R}^n$

2. (Pivoting)
   
   (a) Prove that the matrix
       \[
       \begin{pmatrix}
       0 & 1 \\
       1 & 1
       \end{pmatrix}
       \]
   
   does not have an $LU$ decomposition. **Hint:** assume that such decomposition exists and then show that this brings a contradiction.

   (b) Does the system
       \[
       \begin{pmatrix}
       0 & 1 \\
       1 & 1
       \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}
       \]
   
   have a unique solution for all $a, b \in \mathbb{R}$? (Why?)

   (c) How can you modify the system in part (b) so that $LU$ decomposition applies?

3. (Partial Pivoting)
   
   Consider the linear system, $Ax = b$, where $A$ is the following matrix,
   
   \[
   A = \begin{pmatrix}
   -5 & 2 & -1 \\
   1 & 0 & 3 \\
   3 & 1 & 6
   \end{pmatrix}
   \]

   (a) Using **partial pivoting technique**, determine the $P, L, U$ decomposition of the matrix $A$, such that $PA = LU$. (Show **EACH STEP** in the decomposition.)

   (b) Use the $P, L, U$ decomposition found in (a) to find the solution to
       \[
       Ax = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}
       \]
       (Show **ALL** relevant steps).
(c) Use the \( P, L, U \) decomposition found in (a) to find the solution to
\[
Ax = \begin{pmatrix}
0 \\
1 \\
5
\end{pmatrix}
\]
(Show ALL relevant steps).

4. (Partial Pivoting: MATLAB program)

Write a program to find the \( LU \) decomposition of a given \( n \times n \) matrix \( A \) using partial pivoting. The program should return the updated matrix \( A \) and the pivot vector \( p \). In MATLAB, name your file mylu.m, the first few lines of which should be as follows:

```matlab
function [a,p]=mylu(a)

% [n n]=size(a); p=(1:n)'; (your code here!)
```

The code above sets \( n \) equal to the dimension of the matrix and initializes the pivot vector \( p \). Make sure to store the multipliers \( m_{ij} \) in the proper matrix entries. For more help on function m-files see pages 9–13 of the MATLAB Primer by Kermit Sigmon available from the course webpage. You should experiment with a few small matrices to make sure your code is correct. Check if matrices resulting in \( LU \) decomposition satisfy \( PA = LU \). As a test of your code, in MATLAB execute the statements

```matlab
>>diary mylu.txt
>>format short e
>>type mylu.m
>>a=[2 2 -3;3 1 -2;6 8 1];
>>[a,p]=mylu(a)
>>diary off
```

Print and hand-in the text file containing your program.

5. (a) Consider the matrix
\[
A = \begin{bmatrix}
2 & -3 & 1 \\
-4 & 1 & 2 \\
5 & 0 & 1
\end{bmatrix}.
\]

Compute \( \|A\|_\infty \) and find a vector \( x \) such that \( \|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty \).

(b) Find an example of a \( 2 \times 2 \) matrix \( A \) such that \( \|A\|_\infty = 1 \) but \( \rho(A) = 0 \). This shows that the spectral radius \( \rho(A) = \{ \max |\lambda| : \lambda \text{ is an eigenvalue of } A \} \) does not define a matrix norm.

6. Consider the matrix, right side vector, and two approximate solutions
\[
A = \begin{pmatrix}
1.2969 & 0.8648 \\
0.2161 & 0.1441
\end{pmatrix}, \quad b = \begin{pmatrix}
0.8642 \\
0.1440
\end{pmatrix}, \quad x_1 = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad x_2 = \begin{pmatrix}
0.9911 \\
-0.4870
\end{pmatrix}.
\]
(a) Show that $x = (2, -2)^T$ is the exact solution of $Ax = b$.

(b) Compute the error and residual vectors for $x_1$ and $x_2$.

(c) Find $||A||_\infty, ||A^{-1}||_\infty$ and $\text{cond}_\infty(A)$ (you may use MATLAB for this calculation).

(d) In class we proved a theorem relating the condition number of $A$, the relative error, and the relative residual. Check this result for the two approximate solutions $x_1$ and $x_2$.

7. (LU factorization)

(a) Write a program that takes the output $A$ and $p$ from problem # 4, along with a righthand side $b$, and computes the solution of $Ax = b$ by performing the forward and backward substitution steps. If you are using MATLAB, name your m-file lusolve.m. The first line of your code lusolve.m should be as follows:

```matlab
function x=lusolve(a,p,b)
(your code here!)
```

**Turn in a copy of your code.**

(b) The famous Hilbert matrices are given by $H_{ij} = 1/(i + j - 1)$. The $n \times n$ Hilbert matrix $H_n$ is easily produced in MATLAB using `hilb(n)`. Assume the true solution of $H_n x = b$ for a given $n$ is $x = [1, \ldots, 1]^T$. Hence the righthand side $b$ is simply the row sums of $H_n$, and $b$ is easily computed in MATLAB using $b = \text{sum}(\text{hilb}(n))'$. Use your codes mylu.m and lusolve.m to solve the system $H_n x = b$ for $n = 5, 10, 15, 20$. For each $n$, using the $\infty$-norm, compute the relative error and the relative residual. Discuss what is happening here. You may find it useful to look at the `cond` command in MATLAB.

8. (Iterative Methods: Analysis).

Recall that an $n \times n$ matrix $A$ is said to be *strictly diagonally dominant* if

$$
\sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}| \text{ for } i = 1, \ldots, n.
$$

Note that the strict inequality implies that each diagonal entry $a_{ii}$ is non-zero. Suppose that $A$ is strictly diagonally dominant.

(a) Show that the Jacobi iteration matrix satisfies $||B_J||_\infty < 1$ and, therefore, Jacobi iteration converges in this case.

(b) For a $2 \times 2$ matrix $A$, show that the Gauss-Seidel iteration matrix also satisfies $||B_{GS}||_\infty < 1$ and, hence, Gauss-Seidel iteration converges as well.

The Rockmore Corp. is considering the purchase of a new computer and will choose either the DoGood 174 or the MightDo 11. They test both computers’ ability to solve the linear system

\[
\begin{align*}
34x + 55y - 21 &= 0 \\
55x + 89y - 34 &= 0
\end{align*}
\]

The DoGood 174 computer gives \( x = -0.11 \) and \( y = 0.45 \), and its check for accuracy is found by substitution:

\[
\begin{align*}
34(-0.11) + 55(0.45) - 21 &= 0.01 \\
55(-0.11) + 89(0.45) - 34 &= 0.00
\end{align*}
\]

The MightDo 11 computer gives \( x = -0.99 \) and \( y = 1.01 \), and its check for accuracy is found by substitution:

\[
\begin{align*}
34(-0.99) + 55(1.01) - 21 &= 0.89 \\
55(-0.99) + 89(1.01) - 34 &= 1.44
\end{align*}
\]

Which computer gave the better answer? Why?

Suggested / Additional problems for Math 529 / Phys 528 students:

10. (Special Matrices)

Consider the matrix

\[
\begin{pmatrix}
b & -1 & 0 \\
-1 & 4 & 1 \\
0 & 1 & 5
\end{pmatrix}.
\]

(a) For what values of \( b \) will this matrix be positive definite? (Hint: theorem on page 215 on leading principal submatrices may be useful.)

(b) For what values of \( b \) will this matrix be strictly diagonally dominant? (Recall that an \( n \times n \) matrix \( A \) is said to be strictly diagonally dominant if

\[
\sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}| \quad \text{for } i = 1, \ldots, n.
\]

Note that the strict inequality implies that each diagonal entry \( a_{ii} \) is non-zero.)

11. Consider a linear system with matrix

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}
\]
(a) Write down the iteration matrices \( B_J \) and \( B_{GS} \) for Jacobi’s Method and Gauss–Seidel.

(b) Find the \( l_{\infty} \) norm and spectral radius of the iteration matrix for Jacobi and Gauss-Seidel. (Recall that the spectral radius of a matrix can be calculated by finding the roots of its characteristic polynomial.)

(c) Which of the two iterative methods will converge for an arbitrary starting point \( x^{(0)} \)? Why?

(d) Write a program to calculate and plot the spectral radius of \( B_{\text{SOR}}(\omega) \) for parameter \( \omega \) in the range \((0, 2)\) in increments of 0.01. Provide the code and the plot. Based on inspection of the graph, what value of \( \omega \) will lead to the fastest convergence?

(e) Use the theorem on page 234 of Bradie to calculate analytically the optimal relaxation parameter \( \omega \) for SOR. Does it match the value predicted in Part (d)?

12. Matrix Norms

(a) Prove that if \( ||A|| < 1 \), then

\[
|| (I - A)^{-1} || \geq \frac{1}{1 + ||A||}.
\]

(b) Suppose that \( A \in \mathbb{R}^{n \times n} \) is invertible, \( B \) is an estimate of \( A^{-1} \), and \( AB = I + E \). Show that the relative error in \( B \) is bounded by \( ||E|| \) (using an arbitrary matrix norm).

13. (Cholesky decomposition) (Cholesky decomposition can be used for symmetric positive definite matrices (see pages 215-217 of the textbook).)

(a) Compute the Cholesky decomposition for matrix

\[
\begin{pmatrix}
16 & -28 & 0 \\
-28 & 53 & 10 \\
0 & 10 & 29
\end{pmatrix}
\]

(b) Construct an algorithm to perform forward and backward substitution on the system \( Ax = b \), given a Cholesky decomposition \( A = LL^T \) for the coefficient matrix. How many arithmetic operations are required by the algorithm?

(c) Solve the system \( Ax = b \) with \( b = (8 \ -2 \ 38)^T \) and the above matrix \( A \) by using the Cholesky decomposition and then performing forward and backward substitution.