1. Using the Matlab command qr, find the QR factorization of

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.
\]

For this matrix and with vector \( b = (1 \ 2 \ 3)^T \), using the results of QR factorization, find (i) orthonormal bases for \( \mathcal{R}(A) \) and \( \mathcal{N}(A^T) \), (ii) \( P_A \) and \( P_N \), (iii) vectors \( b_R \) and \( b_N \).

2. Suppose \( A = QR \), where \( Q \) is \( m \times n \) and \( R \) is \( n \times n \). Show that if the columns of \( A \) are linearly independent, then \( R \) must be invertible. [Hint: Study the equation \( Rx = 0 \) and use the fact that \( A = QR \).]

3. Let

\[
x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.
\]

The set \( \{x_1, x_2, x_3\} \) is linearly independent and thus is a basis for a subspace \( W \) of \( \mathbb{R}^4 \). Use the Gram-Schmidt process to construct a set of mutually orthonormal vectors \( q_1, q_2, q_3 \), i.e. an orthonormal basis for \( W \). Use your results to construct a QR factorization of the matrix

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

4. (The purpose of this exercise is to compute the accuracy and efficiency of different methods for QR factorization of a matrix. MATCOM programs are available on the textbook’s web site: http://www.siam.org/books/ot116/)

(a) Compute the QR factorization for each matrix \( A \) of the following data sets as follows:

i. \( [Q, R] = \text{qr}(A) \) from Matlab or \( [Q, R] = \text{housqr}(A) \) or \( \text{housqrn} \) (MATCOM implementations of Householder’s method).

ii. \( [Q, R] = \text{clgrsch}(A) \) (classical Gram-Schmidt implementation from MATCOM)
Method  | $||{(\hat{Q})^T\hat{Q} - I}||_F$  | $||A - \hat{Q}\hat{R}||_F/||A||_F$
---|---|---
housqr  |  | 
clgrsch  |  | 
mdgrsch  |  | 

Table 1: Comparison of different QR factorization methods.

iii. $[Q, R] = \text{mdgrsch}(A)$ from MATCOM (modified Gram-Schmidt implementation from MATCOM)

(b) Using the results of (a), complete the table above for each matrix. $\hat{Q}$ and $\hat{R}$ stand for the computed $Q$ and $R$.

Data set:

i. $A = \text{rand}(25)$,

ii. $A$ is a Hilbert matrix of order 25: $A = \text{hilb}(25)$ in Matlab,

iii. $A = \begin{bmatrix} 1 & 1 & 1 \\ 10^{-4} & 0 & 0 \\ 0 & 10^{-4} & 0 \\ 0 & 0 & 10^{-4} \end{bmatrix}$

iv. a Vandermonde matrix of order 25, that can be generated in Matlab as follows:

```
x=1:25;
A=fliplr(vander(x));
```